On Subspaces of Lp

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On subspaces of L^p

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Introduction

Let $1 \leqq p < 2$. Our main structural result asserts that every subspace of L^p either contains a complemented isomorph of l^p, or imbeds in (is linearly homeomorphic to a subspace of) $L^{p'}$ for some $p' > p$ (Theorem 8). Of course a special case of this result is that every reflexive subspace of L^1 imbeds in L^p for some $p > 1$; a corollary of this special case and previously known results is that every subspace of L^1 contains an unconditional basic sequence (Corollary 12). As further motivation for the content of Theorem 8, we note that for all $1 \leqq p < q \leqq 2$, l^p does not imbed in L^q, while L^q isometrically imbeds in L^p. (See [3] for the real-scalars case of the last mentioned fact and [6] for the case of complex scalars.) Theorem 8 was suggested by recent work of Bretagnolle and Dacunha-Castelle, which shows that every reflexive subspace of L^1 with a symmetric basis, imbeds in L^p for some $p > 1$ [2].

Our main technique (in addition to using the results of [7]), is that of p-absolutely summing operators; we make critical use of this technique for all $p \geqq 2$. However the proof of the special case of our main result, mentioned above, may be accomplished without using this technique. The reader interested mainly in this result may find its proof by reading (the crucial) Lemma 6, Lemma 7, and the first part of the proof of Theorem 8. (He may use as a definition of $I_p(R)$, the smallest constant K satisfying condition 3 of Theorem 1.)

We wish now to indicate in greater detail the results and organization of this paper. In the preliminary Section 1, we give some definitions, notation, and elementary results concerning p-absolutely summing operators defined on $C(S)$-spaces (throughout, "operator" means "bounded linear operator"; "$C(S)$-space" refers to the space of all scalar-valued continuous functions defined on some compact Hausdorff space S). The results of § 1 are all designated as "propositions"; none of the results of § 2 (which begin with Theorem 1) are so designated.

Our main results are contained in § 2. Since L^p isometrically imbeds in L^1 for $p \leqq 2$, we concentrate on studying subspaces of L^1. For R a closed linear subspace of L^1, we define $I_p(R)$ to be the q-absolutely summing norm

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of the canonical map from L^∞ onto R^* (where $1/p + 1/q = 1$). We show that for $p \leqq 2$, $I_p(R)$ is isometrically determined by R, and is equivalent to some intrinsic inequalities concerning R itself (Theorem 1 and Corollary 2). We also obtain there that $I_p(R)$ equals the maximum of the q-absolutely summing norms of all norm-decreasing operators from any $C(S)$-space to R^*. In Lemma 3 we prove that $I_p(l^p) = \infty$ for all $1 < p < 2$ (this important result is due to Schwartz-Kwapien—see [8] and [15]). Lemma 3 also enables us to show that the statement "$I_p(R) < \infty$" is not an isomorphic invariant of R for $p > 2$ (Corollary 3).

We prove in Theorem 5 that $I_2(R) < \infty$ if and only if R is isomorphic to a Hilbert space, while for $1 < p < 2$, $I_p(R) < \infty$ if and only if R imbeds in L^p and l^p does not imbed in R. We also obtain that when $I_p(R) < \infty$, the imbedding of R in L^p can be accomplished in a surprisingly elementary fashion.

The deepest analysis in the paper occurs in Lemma 6; this result yields that if R is a reflexive subspace of L^1, then $\{1 < p < 2\colon I_p(R) < \infty\}$ is a non-empty open interval. The proof of Lemma 6 is quantitative and in a sense, finite-dimensional. As a consequence, there is a version of our main result which has content for finite-dimensional spaces also; this version asserts that subspaces of L^p either uniformly imbed in $L^{p'}$ for some $p' > p$, or contain an almost isometric copy of l_n^p which is the range of an almost contractive projection (for a precise statement, see Theorem 9). In addition to the techniques of the present paper, this version (for the case $p > 1$) leans upon the localization techniques developed in [11]. The case $p = 1$ of Theorem 9 yields that $\mathcal{P}_\lambda$ spaces of sufficiently high dimension contain almost isometric copies of l_n^∞ (Corollary 10). A simple consequence of Theorem 8 is that if X is a reflexive quotient space of some $C(S)$-space, then for some p, $2 \leqq p < \infty$, every operator from every $C(S)$-space to X is p-absolutely summing (and hence X is isomorphic to a quotient space of $L^p(\mu)$ for some measure μ). (Corollary 11.) The reader is referred to [4] for a study of the spaces X such that every operator from a $C(S)$-space to X is 2-absolutely summing. The final result of § 2, Theorem 13, summarizes the consequences of the preceding results concerning subspaces of L^p, $1 < p < 2$.

The paper concludes with an appendix on $p - r$ absolutely summing operators defined on $C(S)$-spaces. The final result shows that for $2 < r < p < \infty$, every operator from a $C(S)$-space to L^p is $p - r$ absolutely summing. (The results of § 1 and the appendix were presented at a conference on $\mathfrak{L}_p$ space theory held at Louisiana State University in June, 1971.)

1. p-absolutely summing operators defined on $C(S)$-spaces

We first give some definitions and notation used throughout. Given X and Y real or complex Banach spaces and $\lambda < \infty$, we say that X is *λ-isomorphic* to Y if there is an invertible operator T from X onto Y with $\|T\| \, \|T^{-1}\| \leqq \lambda$. If X and Y are isomorphic (i.e., linearly homeomorphic) we put $d(X, Y)$ equal to the infimum of the numbers λ such that X is λ-isomorphic to Y. We say that Y is *λ-complemented* in X if $Y \subset X$ and there is a projection (=bounded linear projection) from X onto Y of norm at most λ. We say that Y *imbeds* in X if Y is isomorphic to a subspace of X. S_X denotes the set of x in X with $\|x\| \leqq 1$. X^* denotes the dual of X; when convenient, we denote $x^*(x)$ by $\langle x^*, x\rangle$. Given $1 \leqq p < \infty$, and an operator $T\colon X \to Y$; T is called *p-absolutely summing* (notation: p-a.s.) if there is a constant $K < \infty$; so that for any integer n and n-elements $x_1, \cdots, x_n$ in X,

$$(\textstyle\sum_{i=1}^{n} \|Tx_i\|^p)^{1/p} \leqq K \sup (\textstyle\sum_{i=1}^{n} |x^*(x_i)|^p)^{1/p} , \tag{1}$$

the supremum taken over all $x^* \in X^*$ with $\|x^*\| \leqq 1$. The smallest possible K satisfying (1) for all n and $x_1, \cdots, x_n$ in X, will be called the p-a.s. norm of T.

We begin with a characterization of the p-a.s. norm of operators defined on the spaces l_n^∞ (i.e., on finite-dimensional $C(S)$-spaces). We introduce the following notation and definition: $\{e_1, \cdots, e_n\}$ denotes the natural basis of l_n^∞. A subspace Y of the dual of a Banach space X is called isometrically norming if for all $x \in X$, $\|x\| = \sup_{y \in S_Y} |y(x)|$.

PROPOSITION 1. *Let X be a Banach space, n a positive integer; K a positive number, $1 \leqq q < \infty$, and $1/q + 1/p = 1$. Let $T\colon l_n^\infty \to X$ be a given operator, and let $Te_i = x_i$, $1 \leqq i \leqq n$. Then the following statements are all equivalent:*

1. *T has q-a.s. norm less than or equal to K.*
2. *For all m and matrices (y_{ij}) $(1 \leqq j \leqq n,\ 1 \leqq i \leqq m)$ of scalars,*

$$(\textstyle\sum_{i=1}^{m} \|\sum_{j=1}^{n} y_{ij}x_j\|^q)^{1/q} \leqq K \sup_{1\leqq j\leqq n} (\textstyle\sum_{i=1}^{m} |y_{ij}|^q)^{1/q} . \tag{2}$$

3. *For $\binom{\text{some}}{\text{any}}$ isometrically norming subspace Y of X^*, and for any m and $y_1, \cdots, y_m$ in Y,*

$$\textstyle\sum_{j=1}^{n} (\sum_{i=1}^{m} |\langle y_i, x_j\rangle|^p)^{1/p} \leqq K (\sum_{i=1}^{m} \|y_i\|^p)^{1/p} . \tag{3}$$

Proof. It is easily seen that 1 and 2 are equivalent. Now fixing m, we shall prove that 2 and 3 are equivalent. This is a simple duality argument; we consider the Banach space $(l_m^q \oplus \cdots \oplus l_m^q)_\infty$, the n-fold cartesian product under sup-of-components norm. Its elements may be denoted by matrices (y_{ij}); the norm is then

$$\|(y_{ij})\| = \sup_{1 \leqq j \leqq n} \left(\sum_{i=1}^m |y_{ij}|^q\right)^{1/q} .$$

The dual of this space is then equal to $(l_m^p \oplus \cdots \oplus l_m^p)_1$, the n-fold cartesian product under sum-of-components norm. The elements of the last-mentioned space may be denoted by matrices (α_{ij}) under the norm

$$\|(\alpha_{ij})\| = \sum_{j=1}^n \left(\sum_{i=1}^m |\alpha_{ij}|^p\right)^{1/p} ;$$

the pairing between the two spaces is given by

$$\langle(\alpha_{ij}), (y_{ij})\rangle = \sum_{i,j} \alpha_{ij} y_{ij} .$$

Now to show that $2 \Rightarrow 3$; let $y_1, \cdots, y_m$ be arbitrary elements in X^*, and assume 2: Then for all matrices (y_{ij}),

$$\begin{aligned} \left|\sum_{i,j} y_{ij}\langle y_i, x_j\rangle\right| &= \left|\sum_{i,j} \langle y_i, y_{ij}x_j\rangle\right| \\ &\leqq \left(\sum \|y_i\|^p\right)^{1/p}\left(\sum_i \left\|\sum_j y_{ij}x_j\right\|^q\right)^{1/q} \\ &\leqq K\left(\sum \|y_i\|^p\right)^{1/p} \|(y_{ij})\| . \end{aligned}$$

Thus putting $\alpha_{ij} = (y_i, x_j)$, we have that (α_{ij}) yields a linear functional on the space of (y_{ij})'s of norm at most $K(\sum \|y_i\|^p)^{1/p}$. Hence (3) holds.

Now fix Y an isometrically norming subspace of X^*, and assume (3) holds for all $y_1, \cdots, y_m$ in Y. Fix a matrix (y_{ij}), and let $\varepsilon > 0$. Then we may choose $y_1, \cdots, y_m$ in Y with $\sum_{i=1}^m \|y_i\|^p = 1$ such that

$$\begin{aligned} \left(\sum_{i=1}^m \left\|\sum_{j=1}^n y_{ij}x_j\right\|^q\right)^{1/q} &\leqq \left|\sum_{i,j} \langle y_i, y_{ij}x_j\rangle\right| + \varepsilon \\ &= \left|\sum_{i,j} y_{ij}\langle y_i, x_j\rangle\right| + \varepsilon . \end{aligned} \tag{4}$$

But in turn;

$$\begin{aligned} &\left|\sum_{i,j} y_{ij}\langle y_i, x_j\rangle\right| \\ &\leqq \|(y_{ij})\| \sum_{j=1}^n \left(\sum_{i=1}^m |\langle y_i, x_j\rangle|^p\right)^{1/p} \leqq K\|(y_{ij})\| \end{aligned}$$

by (3) and our natural pairing.

We thus have by (4) and the above, that

$$\left(\sum_{i=1}^m \left\|\sum_{j=1}^n y_{ij}x_j\right\|^q\right)^{1/q} \leqq K\|(y_{ij})\| + \varepsilon$$

for all $\varepsilon > 0$. Hence (2) holds for all matrices (y_{ij}). Q.E.D.

The next result is an easy consequence of the definition of q-a.s. operators; its proof will be omitted.

PROPOSITION 2. *Let S be a compact Hausdorff space, and $\{E_\alpha: \alpha \in \Gamma\}$ a family of finite-dimensional subspaces of $C(S)$ with the following properties*:

(i) *Given $\varepsilon > 0$, n a positive integer and $f_1, \cdots, f_n$ in $C(S)$, there exist α and $e_1^\alpha, \cdots, e_n^\alpha$ in E_α with $\|f_i - e_i^\alpha\| < \varepsilon$ for all i, $1 \leqq i \leqq n$.*

(ii) *For all α, E_α is isometric to l_m^∞ where $m = \dim E_\alpha$.*

Let X be a Banach space, $T: C(S) \to X$ an operator, $K < \infty$, and $1 \leqq q < \infty$.

Then T is q-a.s. with q-a.s. norm less than or equal to K if (and only if) $T \mid E_\alpha$ has q-a.s. norm less than or equal to K, for all α.

For the sake of "symmetry" in the statement of the next result, we recall the

Definition. A map $T\colon X \to Y$ is said to be *p-nuclear* if it admits a factorization of the form

$$\begin{tikzcd} l^\infty \arrow[r, "\Delta"] & l^p \arrow[d, "V"] \\ X \arrow[u, "U"] \arrow[r, "T"] & Y \end{tikzcd}$$

where Δ is a diagonal map. If we require that $\| U \| \leqq 1$ and $\| V \| \leqq 1$, then the infimum of $\| \Delta \|$ over all possible such factorizations is called the p-nuclear norm of the map. (We shall have no use for p-nuclear maps in the sequel.)

The next result is critical for our work in § 2.

PROPOSITION 3. *Let $1 \leqq q < \infty$, $1/p + 1/q = 1$, X a Banach space, and $K < \infty$. Then the following statements are all equivalent:*

1. *For every compact Hausdorff space S and operator $T\colon C(S) \to X$, T is q-a.s. with q-a.s. norm less than or equal to $\| T \| K$.*

2. *For $\left(\begin{smallmatrix}\textit{some}\\ \textit{any}\end{smallmatrix}\right)$ isometrically norming subspace Y of X^*, for all n, $x_1, \cdots, x_n$ in X, and matrices (y_{ij}) $(1 \leqq i \leqq n,\ 1 \leqq j \leqq n)$ of scalars;*

$$\Big(\sum\nolimits_i \Big\| \sum\nolimits_j y_{ij} x_j \Big\|^q\Big)^{1/q} \leqq K \sup\nolimits_j \Big(\sum\nolimits_i | y_{ij} |^q\Big)^{1/q} \sup\nolimits_{y \in S_Y} \sum\nolimits_i | \langle y, x_i \rangle | \,.$$

3. *For $\left(\begin{smallmatrix}\textit{some}\\ \textit{any}\end{smallmatrix}\right)$ isometrically norming subspace Y of X^*, for all n, $x_1, \cdots, x_n$ in X, and $y_1, \cdots, y_n$ in Y,*

$$\sum\nolimits_j \Big(\sum\nolimits_i | \langle y_i, x_j \rangle |^p\Big)^{1/p} \leqq K\Big(\sum\nolimits_i \| y_i \|^p\Big)^{1/p} \sup\nolimits_{y \in S_Y} \sum\nolimits_j | \langle y, x_j \rangle | \,.$$

4. *Every p-nuclear map from X to an arbitrary space B, with p-nuclear norm less than or equal to one, is* 1*-a.s. with* 1*-a.s. norm less than or equal to K.*

Remark. We are indebted to the referee for pointing out that 4 of Proposition 3 is equivalent to the assertion obtained by replacing "p-nuclear" by "p-a.s." (This follows easily from the fact (c.f. [12]) that for any n and any operator T from X to l_n^∞, the p-nuclear and p-a.s. norms of T are equal.)

Proof. All the assertions are simple consequences of the definitions, Propositions 1 and 2, and the observation that if $U\colon l_n^\infty \to X$ is a given operator with $Ue_i = x_i$, for all i, and Y is an isometrically norming subspace of X^*, then

$$\| U \| = \sup_{y \in S_Y} \sum_{i=1}^n | \langle y, x_i \rangle | \, .$$

Thus, it is immediate from the definitions that $1 \Rightarrow 2$. Fixing $x_1, \cdots, x_n$ in x, then by Proposition 1 (or rather, the proof of $2 \Leftrightarrow 3$ of Proposition 1), 2 and 3 of Proposition 3 are equivalent. Since we allow arbitrary n and elements $x_1, \cdots, x_n$ in X, Statement 2 is equivalent to the statement which results from allowing the matrices (y_{ij}) to be arbitrary rectangular rather than square; hence 2 implies (by Proposition 1) that for all n, every operator U from l_n^∞ into X has q-a.s. norm less than or equal to $K \| U \|$. Consequently if S is a given compact Hausdorff space, let $\{E_\alpha : \alpha \in \Gamma\}$ be a family of finite-dimensional subspaces of $C(S)$ satisfying the hypotheses of Proposition 2. (The existence of such a family is well-known; it follows by using partitions of unity.) Then assuming 2 of Proposition 3, and letting $T: C(S) \rightarrow X$, we have that for all α, $T | E_\alpha$ has q-a.s. norm less than or equal to $\| T | E_\alpha \| K$ which is less than or equal to $\| T \| K$, and hence 1 follows by Proposition 2.

It is fairly easy to see that assertions 3 and 4 of Proposition 3 are equivalent. Assuming 3 and letting $x_1, \cdots, x_n$ in X, the inequality in 3 also holds for infinite sequences $y_1, y_2, \cdots$ in Y. Thus suppose that $T: X \rightarrow B$ is a p-nuclear map with p-nuclear norm $\leqq 1$. Then given $\varepsilon > 0$; we can choose $y_1, \cdots, y_n, \cdots$ in X^* with $(\sum \| y_i \|^p)^{1/p} \leqq 1 + \varepsilon$, and a map $V: l^p \rightarrow B$ of norm at most one, so that defining $U: X \rightarrow l^p$ by $Ux = \sum_{i=1}^\infty \langle y_i, x \rangle e_i$ (where $e_1, e_2 \cdots$ are the unit-basis-vectors of l^p), then $T = VU$. Now the left side of the inequality in 3 is equal to $\sum_{j=1}^n \| Ux_j \|$. Hence

$$\begin{aligned} \sum_{j=1}^n \| Tx_j \| &= \sum_{j=1}^n \| VUx_j \| \\ &\leqq \sum_{j=1}^n \| Ux_j \| \leqq K \, (1 + \varepsilon) \sup_{y \in S_{X^*}} \sum_j | \langle y, x_j \rangle | \, , \end{aligned}$$

whence since $\varepsilon > 0$ was arbitrary, T has a.s. norm at most K, hence 4 holds. On the other hand; suppose 4 holds, and suppose $y_1, \cdots, y_n$ in X^* are fixed with $\sum \| y_i \|^p \leqq 1$. Then the map $U: X \rightarrow l^p$ defined by $Ux = \sum_{i=1}^n \langle y_i, x \rangle e_i$ is obviously p-nuclear with p-nuclear norm at most one. Hence given $x_1, \cdots, x_n$ in X;

$$\sum_j \| U(x_j) \| \leqq K \sup_{y \in S_{X^*}} \sum_j | \langle y, x_j \rangle | \, .$$

This means precisely that the inequality of 3 holds provided $\sum \| y_i \|^p \leqq 1$; the fact that the inequality is homogeneous in the n-tuples $(y_1 \cdots y_n)$ shows that it holds in general.

Remarks. 1. Fix the Banach space X, $1 < q < \infty$. It follows easily from the uniform boundedness principle that if there is an infinite-dimensional $\mathfrak{L}_\infty$ space Y (as defined in [9]) such that every operator from Y to X is q-a.s., then there is a $K < \infty$ satisfying the equivalent conditions of Proposition 3. Conversely, if Proposition 3 holds for some $K < \infty$, then every operator from any $\mathfrak{L}_\infty$-space to X is q-a.s. Similarly, Proposition 3 holds for some $K < \infty$, if and only if for all Banach spaces Y, every p-nuclear operator from X to Y is absolutely summing.

2. The only property of $C(S)$ spaces used in the proof of Proposition 3 is that they admit a family of finite-dimensional subspaces $\{E_\alpha : \alpha \in \Gamma\}$ satisfying (i) and (ii) of Proposition 2. It is known that a Banach space B admits such a family if and only if B^* is isometric to $L^1(\mu)$ for some measure μ on a measurable space. (We call such spaces Y, L_1-preduals.) Thus Statement 1 of Proposition 3 is equivalent to the statement obtained by replacing $C(S)$ by an arbitrary L_1-predual, B.

3. It is a consequence of a deep theorem of Dvoretzky that if $q < 2$ (and the scalars are real), then no infinite dimensional Banach space X satisfies the hypotheses of Proposition 3. However it is also well-known that Hilbert space does satisfy Proposition 3 when $q = 2$.

4. For a generalization of Proposition 3 to $p - r$-a.s. operators, see Proposition A1 of the Appendix.

We recall finally the factorization theorem of Pietsch (Theorem 2 of [12]; see also [15]):

PROPOSITION 4. *Let S be a compact Hausdorff space, X a Banach space, $1 \leqq q < \infty$, and $T: C(S) \to X$ a q-a.s. operator with q-a.s. norm equal to K. Then there exists a regular Borel probability measure μ on S, and an operator $V: L^q(\mu) \to X$ with $\|V\| = K$, so that $Tf = Vif$ for all $f \in C(S)$ (where $i: C(S) \to L^q(\mu)$ is the natural map).*

We note that, if there is a probability measure μ on S and an operator $V: L^p(\mu) \to X$ so that $T = Vi$, then the q-a.s. norm of T is less than or equal to $\|V\|$. Finally, Proposition 4 yields immediately the (well-known) fact that if $T: C(S) \to X$ has q-a.s. norm less than or equal to K and $q < q' < \infty$, then also T has q'-a.s. norm less than or equal to K.

2. The main results

Throughout the rest of this paper, μ, ν and λ denote measures on some measurable space $(\Omega, \mathcal{S})$ with μ a *probability* measure (i.e. $\mu(\Omega) = 1$). $L^p(\nu)$ denotes the real or complex space usually denoted $L^p(X, \mathcal{S}, \mu)$; L^p denotes

$L^p(m)$ where m equals Lebesgue measure on the Lebesgue-measurable subsets of the unit interval, and l^p denotes $L^p(\nu)$ where ν is the measure on the positive integers where for all n, $\nu\{n\} = 1$. In reality, we could have restricted ourselves to L^p, since every separable subspace of $L^p(\nu)$ for arbitrary ν isometrically imbeds in L^p. However, in addition to giving information in the non-separable case, considering spaces $L^p(\mu)$ allows for notational convenience in our proofs.

The first theorem of this section provides the foundation for our main results. (Essentially, the new ideas in § 2 may be found in the proofs of Theorem 1, Lemma 6, and Theorem 8.)

THEOREM 1. *Let R be a closed linear subspace of $L^1(\mu)$, $1 < p < \infty$, $1/p + 1/q = 1$, $K < \infty$. Then the following three statements are equivalent:*

1. *For any positive integer n and elements $r_1, \cdots, r_n$ of R,*

$$\int (\sum |r_i|^p(t))^{1/p} d\mu(t) \leqq K(\sum \|r_i\|^p)^{1/p} .$$

2. *The natural map from $L^\infty(\mu)$ onto R^* has q-a.s. norm less than or equal to K.*

3. *There exists a non-negative measurable function ϕ with $\int \phi d\mu \leqq 1$, so that for all $r \in R$, $r(t) = 0$ for (almost) all t belonging to $\{t\colon \phi(t) = 0\}$, and such that*

$$\left(\int |r|^p(t)\phi^{1-p}(t)d\mu(t)\right)^{1/p} \leqq K\int |r(t)|\, d\mu(t) .$$

Moreover, if $1 < p \leqq 2$ and any of these three statements hold, then every operator T from every $C(S)$-space to R^* is q-a.s. with q-a.s. norm less than or equal to $K\|T\|$.

Of course we define the function $|r|^p(t)\phi^{1-p}(t)$ to be equal to zero provided $\phi(t) = 0$. If we put $d\nu = \phi d\mu$ and define $U\colon R \to L^1(\nu)$ by $Ur = r/\phi$ (using the same obvious convention), then U is an isometry, and 3 is simply the statement that

$$\|Ur\|_{L^p(\nu)} \leqq K\|Ur\|_{L^1(\nu)} \qquad \text{for all } r \in R.$$

Proof. $1 \Rightarrow 2$: Let i denote the natural map from $L^\infty(\mu)$ onto R^*. By virtue of Proposition 2, it suffices to show that given any positive integer n and n disjointly supported functions $\phi_1, \cdots, \phi_n$ in L^∞, each of sup-norm 1, and letting E denote their linear span, then $i \mid E$ has q-a.s. norm less than or equal to K. In turn, since R may be regarded as an isometrically norming subspace of R^{**}, it suffices to prove by part 3 of Proposition 1 that given $r_1, \cdots, r_m$ in R,

$$(5)\qquad \sum_{j=1}^{n}\left(\sum_{i=1}^{m} |\langle r_i, \phi_j\rangle|^p\right)^{1/p} \leqq K\left(\sum_{i=1}^{m} \|r_i\|^p\right)^{1/p}$$

(where $\langle\ ,\ \rangle$ denotes the natural pairing between $L^1(\mu)$ and $L^\infty(\mu)$). (Actually, the conclusion of Theorem 1 implies that R must be reflexive, i.e., R equals R^{**}.) Now let $F_1, \cdots, F_n$ be the disjoint measurable sets such that for all i, $\phi_i(t) = 0$ for (almost) all $t \notin F_i$. Then the left side of (5) is equal to

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left|\int_{F_j} r_i(t)\phi_j(t)d\mu(t)\right|^p\right)^{1/p}$$

$$\leqq \sum_{j=1}^{n}\int_{F_j}\left(\sum_{i=1}^{m} |r_i|^p(t)\right)^{1/p}d\mu(t) \leqq \int\left(\sum_{i=1}^{m} |r_i|^p(t)\right)^{1/p}d\mu(t)$$

which is less than or equal to the right side of (5), by hypothesis.

$2 \Rightarrow 3$: We regard $L^\infty(\mu)$ as being equal to $C(S)$ for a certain compact Hausdorff space S; we regard μ as being a regular Borel probability measure on S, and we regard $R \subset L^1(\mu) \subset C(S)^*$. Let j denote the natural quotient map of $C(S)$ onto R^*. Then by Proposition 4 and the hypotheses of 2, there is a regular Borel probability measure ν on S and a map $V: L^q(\nu) \rightarrow R^*$ so that $j = Vi$ where i is the natural map of $C(S)$ into $L^q(\nu)$, with $\|V\| \leqq K$. Of course then R^* is isomorphic to a quotient space of $L^q(\nu)$, a reflexive space, hence R is reflexive. Now we have

$$(6)\qquad j^* = i^*V^* .$$

j^* is nothing but the identity injection of R into $C(S)^*$ (with range contained in $L^1(\mu)$ of course), and i^* is the natural map of $L^p(\nu)$ into $C(S)^*$ defined by $\langle i^*f, \phi\rangle = \int f(s)\phi(s)d\nu(s)$ for all $f \in L^p(\nu)$ and $\phi \in C(S)$. Of course then i^* has its range contained in $L^1(\nu)$. Now by the Radon-Nikodym theorem there is a Borel-measurable and μ-integral non-negative function ϕ, and a measure ν_1 singular with respect to μ, so that $d\nu = \phi d\mu + d\nu_1$; moreover $\int \phi d\mu \leqq 1$ since ν is a probability measure.

It follows by (6) that for all $r \in R$;

$$r d\mu = (V^*r)\phi d\mu + (V^*r)d\nu_1 .$$

Since ν_1 is singular with respect to μ, $(V^*r)d\nu_1 = 0$ for all $r \in R$, and we have the equation $r d\mu = (V^*r)\phi d\mu$, for all $r \in R$. Hence $V^*r = r/\phi$ (a.s.) for all such r. We also know that $\|V^*\| \leqq K$; since

$$\|V^*r\|_{L^p(d\nu)} = \left(\int |V^*r|^p\phi d\mu\right)^{1/p},$$

we thus obtain that

$$\left(\int \frac{|r|^p}{\phi^p}\phi d\mu\right)^{1/p} \leqq \|V^*r\| \leqq K\|r\|,$$

which, after following through the appropriate identification between $C(S)$ and $L^\infty(\mu)$, implies 3.

$3 \Rightarrow 1$: Fixing n and $r_1, \cdots, r_n$ belonging to R,

$$\begin{aligned}\int\left(\sum |r_i|^p(t)\right)^{1/p} d\mu(t) &= \int\left(\frac{\sum |r_i|^p(t)}{\phi^p(t)}\right)^{1/p}\phi(t)d\mu(t)\\ &\leqq \left(\int\frac{\sum |r_i|^p(t)}{\phi^p(t)}\phi(t)d\mu(t)\right)^{1/p}\\ &= \left[\sum\int\frac{|r_i|^p(t)}{\phi^p(t)}\phi(t)d\mu(t)\right]^{1/p}\\ &\leqq K\left(\sum \|r_i\|^p\right)^{1/p},\end{aligned}$$

the second inequality following by Hölder's inequality, the last one by the hypothesis that 3 holds.

To prove the final assertion of Theorem 1, we recall the known fact that there exists a sequence $f_1, f_2, \cdots$ of functions in L^1 which are isometrically equivalent, as a basic sequence, to the unit vectors of l^p. That is, for any n and scalars $c_1, \cdots, c_n$,

$$\int \left|\sum_{i=1}^n c_i f_i(t)\right| dt = \left(\sum_{i=1}^n |c_i|^p\right)^{1/p} \tag{7}$$

(see [3] and [6], and also the remark immediately following this proof). Now we shall apply the criterion given by Statement 3 of Proposition 3. Let $x_1, \cdots, x_n$ in R^* be given, with

$$\sum_{j=1}^n |\langle y, x_j\rangle| \leqq \|y\| \qquad \text{for all } y \in R\,. \tag{8}$$

Let $y_1, \cdots, y_n$ be given elements of R. Then

$$\begin{aligned}&\sum_j \left(\sum_i |\langle y_i, x_j\rangle|^p\right)^{1/p}\\ &= \sum_j \int_0^1 \left|\sum_i f_i(t)\langle y_i, x_j\rangle\right| dt && \text{by (7)}\\ &= \int_0^1 \sum_j \left|\left\langle \sum_i f_i(t) y_i, x_j\right\rangle\right| dt\\ &\leqq \int_0^1 \left\|\sum f_i(t) y_i\right\| dt && \text{by (8)}\\ &= \int_\Omega\int_0^1 \left|\sum f_i(t) y_i(w)\right| dt\, d\mu(w) && \text{by Fubini's theorem}\\ &= \int_\Omega \left(\sum |y_i(w)|^p\right)^{1/p} d\mu(w) && \text{by (7)}\\ &\leqq K\left(\sum \|y_i\|^p\right)^{1/p} && \text{by condition 1 of Theorem 1.}\end{aligned}$$

This completes the proof of Theorem 1.

Remark. It follows from the results of [6] that complex l^p isometrically imbeds in complex L^q for $1 \leqq q \leqq p \leqq 2$. We wish to sketch an alternate argument for this fact, which shows that fixing p, there is a certain sequence of complex-valued independent random variables which accomplishes this imbedding simultaneously for all $1 \leqq q < p$.

Given a complex-valued measurable function f defined on Ω, its distribution is the probability measure ν defined on the Borel subsets of the complex numbers C by $\nu(E) = \mu\{z: f(z) \in E\}$ and its characteristic function $\hat{\nu}$ is defined by $\hat{\nu}(z) = \int_{\mathrm{C}} e^{i\mathcal{R}e z\bar{\lambda}} d\nu(\lambda)$, $\left(= \int_{\Omega} e^{i\mathcal{R}e z\bar{f}(t)} d\mu(t)\right)$. It is known that ν is uniquely determined by $\hat{\nu}$ and $\int_{\Omega} |f|^p d\mu = \int_{\mathrm{C}} |z|^p d\nu(z)$. It is also known that fixing $1 < p \leqq 2$, there exists a probability measure ν on C such that $\hat{\nu}(z) = e^{-|z|^p}$. This is equivalent to the assertion that $e^{-|x^2+y^2|^{p/2}}$ is a positive definite function of the two real-variables x and y. In fact, there is a positive function p so that $d\nu(re^{i\theta}) = p(r)drd\theta$. It follows that

$$\hat{\nu}(z) = \int_0^{2\pi}\int_0^{\infty} \cos\,(\mathcal{R}e\, zre^{-i\theta})p(r)drd\theta\ .$$

Now fixing q, $1 \leqq q \leqq 2$, there exists a positive constant C_q so that for all z,

$$C_q\,|z|^q = \int_0^{2\pi}\int_0^{\infty} \frac{1 - \cos \mathcal{R}e\,(zre^{-i\theta})}{r^{1+q}} drd\theta\ .$$

Hence we have that

$$\begin{aligned}\int C_q\,|z|^q d\nu(z) &= \int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty} \frac{1 - \cos \mathcal{R}e\, te^{i\phi}re^{-i\theta}}{r^{1+q}} p(t)drd\theta dtd\phi \\ &= \int_0^{2\pi}\int_0^{\infty} \frac{1 - \hat{\nu}(re^{i\theta})}{r^{1+q}} drd\theta \\ &= 2\pi\int_0^{\infty} \frac{1 - e^{-r^p}}{r^{1+q}} dr\ ,\end{aligned}$$

the second equality holding by Tonelli's theorem. We thus have that $\int |z|^q d\nu(z) < \infty$ if and only if $\int_0^{\infty} (1 - e^{-r^p})/(r^{1+q})dr < \infty$ which occurs if and only if $q < p$. Now let $f_1, f_2, \cdots$ be complex-valued independent random variables defined on $[0, 1]$ so that for all j,

$$\int e^{i\mathcal{R}e z\overline{f_j(t)}} dt = e^{-|z|^p}\ .$$

Then fixing $q < p$, $f_j \in L^q$; given n and scalars $c_1, \cdots, c_n$, the characteristic function of $\sum c_i f_i$ is equal to

$$e^{-|(\Sigma|c_i|^p)^{1/p} z|^p}$$

(by the independence of the f_i's) which implies that $\|\sum c_i f_i\|_q = (\sum |c_i|^p)^{1/p}\|f_1\|_q$. Thus the closed linear span of the f_j's in L^q is isometric to l^p for all $1 \leqq q < p \leqq 2$.

Definition. Given R a closed linear subspace of $L^1(\mu)$ and $1 < p < \infty$, we define $I_p(R)$ to be the $p/(p-1)$-a.s. norm of the natural map from $L^\infty(\mu)$ onto R^* (provided this map is $p/(p-1)$-a.s.; otherwise we put $I_p(R) = \infty$).

Of course by Theorem 1, $I_p(R)$ is the smallest K satisfying any of the equivalent conditions 1–3 of Theorem 1. In particular, it will be important for our later work to note that if $I_p(R) < \infty$ and ϕ is an integrable non-negative function with $\int \phi d\mu = 1$, then

$$\sup_{r \in S_R} \left(\int \frac{|r|^p}{\phi^p} \phi d\mu \right)^{1/p} \geqq I_p(R) .$$

The next result follows immediately from known results, the work of § 1, and Theorem 1.

COROLLARY 2. *Let R be a closed linear subspace of $L^1(\mu)$ and let p, p' satisfy $1 < p \leqq p' < \infty$; then $I_p(R) \leqq I_{p'}(R)$. If R is isomorphic to a Hilbert space, then $I_2(R) < \infty$. If $I_p(R) < \infty$ for some $p \geqq 2$, then R is isomorphic to a Hilbert space. If $1 < p \leqq 2$, then $I_p(R)$ equals the minimum of the numbers K, such that for all compact Hausdorff spaces S and operators $T: C(S) \rightarrow R^*$, T has $p/(p-1)$-a.s. norm $\leqq \|T\| K$. Consequently if $\tilde{R}$ and R are isomorphic subspaces of $L^1(\mu)$ then $I_p(\tilde{R}) \leqq d(R, \tilde{R}) I_p(R)$.*

Proof. Letting $1/q' + 1/p' = 1 = 1/q + 1/p$, then assuming $I_{p'}(R) < \infty$, the natural map of L^∞ into R^* is q'-a.s., and hence also it is q-a.s. by Proposition 4, since $q' < q$; moreover the same result shows that $I_p(R) \leqq I_{p'}(R)$.

It is known that every operator from a $C(S)$-space to a Hilbert space is 2-a.s. (c.f. [9] and [15]), hence the second statement of the corollary follows. If $I_p(R) < \infty$ for some $p \geqq 2$, then $I_2(R) < \infty$, hence the natural map of L^∞ into R^* is 2-a.s. which implies that R^* and hence R, is isomorphic to a Hilbert space. The last two statements follow easily from the "moreover" part of Theorem 1 and the definition of $I_p(R)$. Q.E.D.

It follows from the last two statements of the corollary, that if $1 < p \leqq 2$ and R is a subspace of $L^1(\mu)$, then $I^p(R)$ is determined solely by the isometry-type of R and not the particular way in which R is imbedded in $L^1(\mu)$. Hence we can speak unambiguously of $I_p(l_n^p)$, for example.

Remark. Actually, the proof of Theorem 1 shows that for $1 < p \leqq 2$, and R a subspace of $L^1(\mu)$, then $I_p(R)$ may be obtained as follows: let $f_1, f_2, \cdots$

be a fixed sequence in L^1 satisfying (7) for all n. Then $I_p(R)$ is the smallest constant K such that for all n and elements $r_1, \cdots, r_n$ in R,

$$\int_0^1 \|\sum f_i(t) r_i\| \, dt \leqq K(\sum \|r_i\|^p)^{1/p} .$$

This shows directly that $I_p(R)$ is isometrically determined by R for $1 < p \leqq 2$. It is also important for our later work, to note that fixing $1 < p < \infty$ and $R \subset L^1(\mu)$, then $I_p(R)$ is determined by the finite-dimensional subspaces of R. In fact, $I_p(R) = \sup I_p(E)$, the supremum taken over all finite-dimensional subspaces E of R.

The next lemma yields that the statement "$I_p(R) < \infty$" is not determined by the isomorphism type of R for $p > 2$, and also shows the result of Schwartz-Kwapien (c.f. [8] and [15]) that $I_p(l^p) = \infty$ for all $1 < p < 2$. (Our argument uses the three series-criterion, as does the argument of Schwartz-Kwapien.)

LEMMA 3. *Let $x_1, x_2, \cdots$ be a sequence of integrable independent identically distributed real-valued random variables defined on* $[0, 1]$, *let R denote their closed-linear span in L^1 over the real scalars and let $1 < p < \infty$. Then $\int |x_1|^p dt < \infty$ if and only if*

$$\lim_{n\to\infty} \sup \int_0^1 (\sum_{i=1}^n |c_i x_i|^p)^{1/p} dt < \infty , \tag{9}$$

the supremum taken over all scalars $c_1, \cdots, c_n$ with

$$(\sum_{i=1}^n |c_i|^p)^{1/p} \leqq 1 .$$

In particular if $1 < p < 2$ and $f_1, f_2, \cdots$ is a sequence of independent functions satisfying (7) *for all n and scalars $c_1, \cdots, c_n$, then putting*

$$r_n^p = \sup_{(\Sigma |c_i|^p)^{1/p} \leqq 1} \int_0^1 (\sum_{i=1}^n |c_i f_i|^p)^{1/p} dt ,$$

$$I_p(l_n^p) \geqq r_n^p \quad \text{and} \quad r_n^p \to \infty \quad \text{as} \quad n \to \infty .$$

Proof. If $\int |x_1|^p dt < \infty$, then

$$\int (\sum |c_i x_i(t)|^p)^{1/p} dt \leqq \left(\int \sum |c_i|^p |x_i(t)|^p dt\right)^{1/p}$$
$$= (\sum |c_i|^p)^{1/p} \left(\int |x_1|^p dt\right)^{1/p}$$

by Hölder's inequality and the fact that the x_i's are identically distributed. Now assume that $\int |x_1|^p dt = \infty$. To show that the left side of (9) is infinite, it suffices to show that there exists a sequence $c_1, c_2, \cdots$ with $\sum |c_i|^p < \infty$, yet such that $\sum |c_i|^p |x_i|^p(t)$ does not converge a.e. (which by the zero-one law, means that $\sum |c_i|^p |x_i|^p(t) = \infty$ a.e.). Let μ be the Borel measure on the real line which equals the distribution of x_1, i.e., $\mu(E) = m\{t: x_1(t) \in E\}$ for all Borel sets E (where m denotes Lebesgue measure on $[0, 1]$). Fix a sequence (c_i) with $c_i \neq 0$ for all i, and define $\tilde{x}_i$ by $\tilde{x}_i(t) = |c_i|^p |x_i(t)|^p$ if $|c_i|^p |x_i(t)|^p \leqq 1$; $\tilde{x}_i(t) = 0$ otherwise. If $\sum |c_i|^p |x_i|^p(t)$ converges a.e., then by the three series criterion of Kolmogorov,

$$\sum \int \tilde{x}_i(t)dt < \infty .$$

For each i,

$$\int \tilde{x}_i(t)dt = |c_i|^p \int_{\{\lambda: |\lambda| \leq 1/|c_i|\}} |\lambda|^p d\mu(\lambda) .$$

Since $\int |x_1|^p dt = \infty$, $\int |\lambda|^p d\mu(\lambda) = \infty$, whence if $c_i \to 0$, then

$$\lim_{i\to\infty} \int_{\{\lambda: |\lambda| \leq 1/|c_i|\}} |\lambda|^p d\mu(\lambda) = \infty .$$

It therefore follows that we may choose non-zero c_i's so that $\sum |c_i|^p < \infty$, yet $\sum \int \tilde{x}_i(t)dt = \infty$, whence $\sum |c_i|^p |x_i|^p(t)$ fails to converge a.e.

To prove the final statement of the lemma, since the quantities r_n^p are isometrically invariant (c.f. the second remark immediately following this proof), it suffices to consider a particular sequence $f_1, f_2, \cdots$ satisfying (7). Let $f_1, f_2, \cdots$ be a sequence of identically distributed independent functions defined on $[0, 1]$, each of L^1-norm one, such that there is a constant c (uniquely determined by the requirement that $\int |f_1| \, dt = 1$) so that

$$\int_0^1 e^{itf_1(x)}dx = e^{-c|t|^p} \qquad \text{for all } t .$$

(Such functions are called stable random-variables of exponent p.) It follows that the f_i's satisfy (7), and also that $\int |f_1|^p dt = \infty$, by the same argument given in the remarks following Theorem 1. Thus $r_n^p \to \infty$ as $n \to \infty$; it is immediate from the definitions that $I_p(l_n^p) \geqq r_n^p$. Q.E.D.

Remarks 1. Let $x_1, x_2, \cdots, R$ and p be as in the statement of Lemma 3. Then if $\int |x_1|^p dt = \infty$, $I_p(R) = \infty$. However, the converse is false for $1 < p < 2$. Indeed by Theorem 8 below, if $I_p(R) < \infty$, then there is an $\varepsilon > 0$ such that $I_{p+\varepsilon}(R) < \infty$, whence $\int |x_1|^{p+\varepsilon} dt < \infty$ by Lemma 3. (It is known that if the x_n's are each of mean zero, then they form a symmetric basis for

R (in fact their closed linear span is isomorphic to an Orlicz sequence space; c.f. [2] and Theorem 2 of [4]).)

2. As pointed out by Schwartz (c.f. [15]), a more detailed analysis of the f_i's (as defined in the above argument) shows that

$$\sum |c_i|^p \, |f_i(t)|^p < \infty \text{ a.e.}$$

if and only if

$$\sum |c_i|^p\left(1 + \log \frac{1}{|c_i|^p}\right) < \infty \, ;$$

it can be deduced from this that

$$(\log n)^{1/p} = 0\left(\frac{1}{n^{1/p}} \int \left(\sum |f_i|^p\right)^{1/p} dt\right) \qquad \text{as } n \to \infty .$$

Hence for $1 < p < 2$, there exists a positive constant c_p so that $r_n^p \geqq c_p (\log n)^{1/p}$ for all n. It is conceivable that $(\log n)^{1/p}$ is the correct order of magnitude for r_n^p; we suspect that I_n^p has a larger order of magnitude than r_n^p. We also note that if $\bar{r}_n^p$ denotes the quantity analogous to r_n^p, defined over the complex field, then it can easily be deduced from the above considerations that $(\log n)^{1/p} = 0(\bar{r}_n^p)$ as well. (It should perhaps be pointed out that r_n^p is isometrically determined; to prove this one considers the expression

$$\iint \left|\sum a_j f_j(w) f_j'(t)\right| dt dw$$

where (f_j') is another sequence satisfying (7) for all n and scalars $c_1, c_2, \cdots, c_n$.)

The next result shows that for subspaces R of L^1 isomorphic to Hilbert space, whether or not $I_p(R) < \infty$ depends on the way in which R is imbedded in L^1, for $p > 2$.

COROLLARY 4. *Given any subinterval I of $[2, \infty)$ containing 2, there exists an infinite-dimensional closed linear subspace R of L^1 such that $I = \{p \geqq 2 \colon I_p(R) < \infty\}$.*

Proof. The complex case is easily deduced from the real one, so we consider only real scalars. Let $x_1, x_2, \cdots$ be a sequence of independent identically distributed square-integrable real valued random variables. It follows from known results and Lemma 3 that

$$\{p \geqq 2 \colon I_p(R) < \infty\} = \left\{p \colon \int |x_1|^p dt < \infty\right\} .$$

The required examples may now be easily constructed.

Remarks. Theorem 8 below shows that if R is a reflexive subspace of

L^1 with R non-isomorphic to Hilbert space, then $\{p > 1: I_p(R) < \infty\}$ is a non-empty open interval contained in $[1, 2]$.

However, given any subinterval I of $[1, 2]$ containing 1, there exists a closed linear subspace R of L^1 such that $I = \{p \leqq 2: R \text{ imbeds in } L^p\}$. Indeed, let $1 < p \leqq 2$ and $x_1, x_2, \cdots$ be a sequence of identically distributed independent symmetric random variables such that for all M sufficiently large,

$$m\{t: |x_1(t)| \geqq M\} = \int_M^\infty \frac{\log t}{t^{1+p}} dt .$$

Then if R denotes the closed linear span of the x_i's in L^1, it can be proved (using the proof of Theorem 2 of [14]) that $I_q(R) < \infty$ for all $q < p$, yet R does not imbed in L^p. Of course, isomorphic imbeddings of l^p in L^1 yield examples where the interval in question is closed. In this connection, we would like to mention that it can be proved that if $1 < p < 2$ is fixed, $c = (2/p)^{1/p}$, and $x_1, x_2, \cdots$ is a sequence of independent random variables such that for all i and Borel sets E,

$$m\{t: x_i(t) \in E\} = \int_{E \cap \{r: |r| \geqq c\}} \frac{1}{|t|^{1+p}} dt ,$$

then the closed linear span of the x_i's in L^r, is isomorphic to l^p for all $1 \leqq r < p$.

Our next result summarizes the isomorphic consequences of Theorem 1. Its proof is an easy consequence of known results and Theorem 1.

THEOREM 5. *Let $1 < p \leqq 2$, $1/p + 1/q = 1$, and R a closed linear subspace of $L^1(\mu)$ of infinite dimension. Then the following statements are all equivalent*:

1. $I_p(R) < \infty$.

2. *R imbeds in $L^p(\mu)$ and l^p does not imbed in R if $p < 2$; if $p = 2$, R is isomorphic to Hilbert space (i.e., R imbeds in $L^2(\mu)$).*

3. *There exists an integrable function ϕ, with $\phi(t) > 0$ for all $t \in \Omega$, so that*

$$\int |r|^p(t)\phi^{1-p}(t)d\mu(t) < \infty \qquad \text{for all } r \in R .$$

4. *For any sequence $r_1, r_2, \cdots$ in R such that $\sum \|r_i\|^p < \infty$,*

$$\int \left(\sum_{i=1}^\infty |r_i|^p(t)\right)^{1/p} d\mu(t) < \infty .$$

5. *There is a surjective bounded linear map from some $C(S)$-space onto R^* which is q-a.s.*

6. *Every bounded linear map from every $C(S)$ space into R^* is q-a.s.*

Proof. It is most convenient to show that all the statements are equivalent to the first. The fact that the first statement implies all the others follows trivially from Theorem 1 and the fact that $I_p(l^p) = \infty$ for $1 < p < 2$ (proved in Lemma 3).

$2 \Rightarrow 1$: Let $\tilde{R}$ be a subspace of $L^p(\mu)$, which is isomorphic to R. Then l^p does not imbed in $\tilde{R}$, hence by the results of [7], there exists a C and an ε, $0 < \varepsilon < 1$, so that for all g in $\tilde{R}$ of norm one,

$$\int_{\{t:|g(t)|\leqq C\}} |g|^p d\mu \geqq \varepsilon .$$

But then for such g,

$$\varepsilon \leqq \int_{\{t:|g(t)|\leqq C\}} |g|^p d\mu \leqq C^{p-1} \int |g|\, d\mu ;$$

thus letting $\tilde{\tilde{R}}$ denote the space $\tilde{R}$ in the L^1-norm, $\tilde{\tilde{R}}$ is a closed linear subspace of L^1 isomorphic to R, and satisfying Condition 3 of Theorem 1 for $K = C^{p-1}/\varepsilon$ and $\phi \equiv 1$. Hence $I_p(\tilde{\tilde{R}}) < \infty$ whence by Corollary 2, $I_p(R) < \infty$.

$3 \Rightarrow 1$: By multiplying by a constant, we may assume that $\int \phi d\mu = 1$. Putting $d\nu = \phi d\mu$ and defining U: $R \rightarrow L^1(\nu)$ by $Ur = r/\phi$, we have that U is an isometry and $UR \subset L^p(\nu)$. It follows by the closed graph theorem that the $L^p(\nu)$ and $L^1(\nu)$ norms on UR are equivalent, which implies Condition 3 of Theorem 1.

$4 \Rightarrow 1$: It is possible to show by a Banach-space argument that Condition 1 of Theorem 1 holds for some $K < \infty$. We may prove this, however, by the following elementary considerations: If this condition fails for all $K < \infty$, then for each positive integer n, we may choose $r_1^n, \cdots, r_{m_n}^n$ in R so that

$$\left(\sum_{i=1}^{m_n} \|r_i^n\|^p\right)^{1/p} \leqq \frac{1}{2^n} ,$$

yet

$$\int \left(\sum_{i=1}^{m_n} |r_i^n|^p(t)\right)^{1/p} d\mu(t) \geqq n .$$

It then follows immediately that $\sum_{n=1}^{\infty} \sum_{i=1}^{m_n} \|r_i^n\|^p < \infty$, yet

$$\int \left(\sum_{n=1}^{\infty} \sum_{i=1}^{m_n} |r_i^{m_n}|^p(t)\right)^{1/p} d\mu(t) \geqq N$$

for all positive integers N, contradicting 4.

Since trivially $6 \Rightarrow 5$, it remains to show that $5 \Rightarrow 1$: By 5 and Proposition 4, there is a regular probability Borel measure ν on S and a map V: $L^q(\nu) \rightarrow R^*$ so that Vi maps $C(S)$ onto R^* where i is the natural map of

$C(S)$ into $L^q(\nu)$. It follows that R is reflexive and $i^* V^*$ is an isomorphic imbedding of R into $C(S)^*$. Since i^* is the natural map of $L^p(\nu)$ into $C(S)^*$ and V^* is also an isomorphism, it follows that $\tilde{R}$ is isomorphic to R, where $\tilde{R}$ denotes the range of $i^* V^*$. Moreover there is a constant $K < \infty$ so that $\|r\|_{L^p(\nu)} \leqq K \|r\|_{L^1(\nu)}$ for all $r \in \tilde{R}$. Thus $I_p(\tilde{R}) < \infty$ by Theorem 1, and hence $I_p(R) < \infty$ by Corollary 2. Q.E.D.

Remark. Let $1 < p < 2$ and $r < p$. Since L^p imbeds in L^r and l^r does not imbed in L^p, we obtain that $I_r(L^p) < \infty$. It can be shown that there is a constant c_p depending only on p, so that $I_r(L^p) \leqq c_p(1/(p-r))^{1/r}$. We conjecture that this is the correct order of magnitude, i.e., that $(1/(p-r))^{1/r} = 0(I_r(L^p))$ as $r \to p$. Incidentally, Theorem 5 implies the result of Kwapien [8] that every map from every $C(S)$ space into L^q is s-a.s., for all $2 < q < s$.

We now pass to a deeper investigation of the structure of subspaces of L^1. The following lemma is critical to our considerations.

LEMMA 6. *Let $1 \leqq p_0 < 2$ and R a closed linear subspace of $L^1(\mu)$ such that $I_p(R) = \infty$ for all $p > p_0$. Then for all k and $\varepsilon > 0$, there exist k elements $r_1, \cdots, r_k$ in R satisfying*

$$(\Delta) \qquad \begin{cases} \|r_i\| = 1 \textit{ for all } i, \textit{ and for all scalars } c_1, \cdots, c_k\,, \\ \|\sum c_i r_i\| \geqq (1-\varepsilon)(\sum |c_i|^{p_0})^{1/p_0}\,. \end{cases}$$

Proof. Fix k and ε with $\varepsilon < 1$. Choose $p > p_0$ so that

$$(10) \qquad (\textstyle\sum_{j=1}^k |c_j|^p)^{1/p} \geqq \sqrt{1-\varepsilon}(\sum_{j=1}^k |c_j|^{p_0})^{1/p_0} \qquad \text{for all scalars } c_1, \cdots, c_k\,.$$

Now choose δ, $0 < \delta < 1$, so that

$$(11) \qquad \delta - (1-\delta^p)^{1/p} k^{(p-1)/p} \geqq \sqrt{1-\varepsilon}\,.$$

Letting N be a large number, we define the following quantities: K is defined by $K^{p-1} = N^{p/2}$ and $\bar{\varepsilon} = K^{-1/2}$.

Now choose $\bar{N}$ so large that for all $N \geqq \bar{N}$,

$$(12) \qquad (1-\bar{\varepsilon})^{p-1}\left[1 - \left(\frac{k}{\bar{\varepsilon} K}\right)^{p-1}\right] - \frac{K^{p-1}}{N^p} \geqq \delta^p\,.$$

(Of course, then automatically, $k/K < 1$.)

Now we may choose a finite-dimensional subspace $\tilde{Y}$ of R, so that putting $N = I_p(Y)$, then $N \geqq \bar{N}$. By Theorem 1, we may choose an integrable non-negative function ϕ, with $\int \phi d\mu = 1$, so that for all $y \in \tilde{Y}$, $\{t: y(t) = 0\}$ is contained, up to a set of measure zero, in $\{t: \phi(t) = 0\}$, and such that

$$\left(\int \frac{|y|^p}{\phi^p} \phi d\mu\right)^{1/p} \leqq N \|y\| \qquad \text{for all } y \in \tilde{Y}\,.$$

Now let ν be the measure defined by $d\nu = \phi d\mu$ and let Y be the subspace of $L^1(\nu)$ defined by $Y = \{y/\phi : y \in \tilde{Y}\}$.

Then Y is isometric to $\tilde{Y}$. Using the notation

$$\|y\|_1 = \int |y|\,d\nu;\ \|y\|_p = \left(\int |y|^p d\nu\right)^{1/p},$$

we have, by Theorem 1, that

$$I_p(Y) = N = \sup_{\substack{y \in Y \\ y \neq 0}} \|y\|_p / \|y\|_1\ ; \tag{13}$$

and thus if ψ is any function with $\psi(t) > 0$ for all $t \in \Omega$, and $\int \psi d\nu = 1$, then there exists a $y \in Y$ so that

$$\int |y|^p \psi^{1-p} d\nu \geqq N^p \quad \text{and} \quad \|y\|_1 = 1\,. \tag{14}$$

Using the above relationships, we shall now prove the following

SUBLEMMA. *There exist disjoint measurable subsets* $E_1, \cdots, E_k$, *and functions* $g_1, \cdots, g_k \in Y$, *so that for all* i; $|g_i| \geqq K$ *on* E_i,

$$\|g_i\| = 1\,, \quad \textit{and} \quad \int_{E_i} |g_i|^p d\nu \geqq \delta^p N^p\,.$$

The lemma follows from the sublemma; i.e., setting $r_i = \phi \cdot g_i$, then $r_1, \cdots, r_k$ satisfy (Δ). To see this, let $\tilde{g}_i = g_i \cdot \chi_{E_i}$ for all i; then $\|g_i - \tilde{g}_i\|_p \leqq N(1-\delta^p)^{1/p}$. Hence

$$\left\|\sum c_i(g_i - \tilde{g}_i)\right\|_p \leqq \left(\sum |c_i|^p\right)^{1/p} N(1-\delta^p)^{1/p} k^{(p-1)/p}\,,$$

for any scalars $c_1, \cdots, c_k$. Thus

$$\begin{aligned} \left\|\sum c_i g_i\right\|_p &\geqq \left\|\sum c_i \tilde{g}_i\right\|_p - \left\|\sum c_i(g_i - \tilde{g}_i)\right\|_p \\ &\geqq \left(\sum |c_i|^p\right)^{1/p} N[\delta - (1-\delta^p)^{1/p} k^{(p-1)/p}] \\ &\geqq \left(\sum |c_i|^p\right)^{1/p} N\sqrt{1-\varepsilon} \qquad \text{by (11)}\,. \end{aligned} \tag{15}$$

But by (13); $\left\|\sum c_i g_i\right\|_p \leqq N \left\|\sum c_i g_i\right\|_1$, hence

$$\left\|\sum c_i g_i\right\|_1 \geqq (\sqrt{1-\varepsilon})\left(\sum |c_i|^p\right)^{1/p} \geqq (1-\varepsilon)\left(\sum |c_i|^{p_0}\right)^{1/p_0} \qquad \text{by (10)}\,.$$

This proves (Δ) in virtue of the fact that $g \to \phi \cdot g$ is an isometry between Y and $\tilde{Y}$. We pass now to the proof of the sublemma, which will be established by induction. (The fact that $|g_i| \geqq K$ on E_i will be important for the proof, although it was not used in the above reasoning.)

To begin the proof, choose $g_1 \in Y$ so that $\|g_1\|_p = N$ and $\|g_1\|_1 = 1$, and let $E_1 = \{t : |g_1(t)| \geqq K\}$. Then $\int_{\sim E_1} |g_1|^p d\nu \leqq K^{p-1}$,

$$\int_{E_1} |g_1|^p d\nu \geqq N^p - K^{p-1} \geqq \delta^p N^p \qquad \text{by (12)}\,.$$

Now suppose $1 \leqq j < k$, $g_1, \cdots, g_j$, and $E_1, \cdots, E_j$ have been chosen, satisfying the conclusion of the sublemma. Then $\nu(\sim \bigcup_{i=1}^j E_i) > 0$; indeed $\nu(E_i) \leqq 1/K$ for all i, hence

$$\nu(\bigcup_{i=1}^j E_i) \leqq \frac{j}{K} \leqq \frac{k}{K} < 1 \qquad \text{by (12)} .$$

Now let

$$\psi = \frac{\bar{\varepsilon}}{j} \sum_{i=1}^j |g_i| \chi_{E_i} + c\chi_{\sim \cup_{i=1}^j E_i} \tag{16}$$

where

$$c = \frac{1 - \dfrac{\bar{\varepsilon}}{j} \sum_{i=1}^j \displaystyle\int_{E_i} |g_i| d\nu}{\nu(\sim \bigcup_{i=1}^j E_i)} .$$

Then ψ is everywhere positive, $\int \psi d\nu = 1$, and moreover

$$c \geqq 1 - \bar{\varepsilon} . \tag{17}$$

Thus by (14), we may choose a function $g_{j+1} \in Y$ so that $\int |g_{j+1}|^p \psi^{1-p} d\nu \geqq N^p$ and $\|g_{j+1}\|_1 = 1$.

Now by (16), if $t \in \bigcup_{i=1}^j E_i$, then

$$\psi^{1-p}(t) \leqq \left(\frac{k}{\bar{\varepsilon}K}\right)^{p-1} \qquad (\text{since } \textstyle\sum |g_i| \chi_{E_i} \geqq K \text{ on } \bigcup_{i=1}^j E_i) , \tag{18}$$

hence

$$\int_{\cup_{i=1}^j E_i} |g_{j+1}|^p \psi^{1-p} d\nu \leqq \left(\frac{k}{\bar{\varepsilon}K}\right)^{p-1} \int |g_{j+1}|^p d\nu \leqq N^p \left(\frac{k}{\bar{\varepsilon}K}\right)^{p-1} . \tag{19}$$

Therefore

$$\begin{aligned} \int_{\sim \cup_{i=1}^j E_i} |g_{j+1}|^p c^{1-p} d\nu &= \int_{\sim \cup_{i=1}^j E_i} |g_{j+1}|^p \psi^{1-p} d\nu \\ &\geqq \left[1 - \left(\frac{k}{\bar{\varepsilon}K}\right)^{p-1}\right] N^p \end{aligned} \tag{20}$$

or

$$\int_{\sim \cup_{i=1}^j E_i} |g_{j+1}|^p d\nu \geqq (1 - \bar{\varepsilon})^{p-1} \left[1 - \left(\frac{k}{\bar{\varepsilon}K}\right)^{p-1}\right] N^p \tag{21}$$

by (20) and (17).

Now put $E_{j+1} = \{x: g_{j+1}(x) \geqq K\} \cap \sim \bigcup_{i=1}^j E_i$. Then

$$\int_{\sim E_{j+1}\cap\sim\cup_{i=1}^{j}E_i} |g_{j+1}|^p d\nu \leqq K^{p-1}\int |g_{j+1}|\, d\nu = K^{p-1}\,; \tag{22}$$

hence

$$\begin{aligned}\int_{E_{j+1}} |g_{j+1}|^p d\nu &\geqq (1-\bar{\varepsilon})^{p-1}\left[1-\left(\frac{k}{\bar{\varepsilon}K}\right)^{p-1}\right]N^p - K^{p-1} \\ &\geqq \delta^p N^p \qquad \text{by (21), (22), and (12)}\,. \quad \text{Q.E.D.}\end{aligned} \tag{23}$$

Remarks 1. The proof of Lemma 6 involves only finite-dimensional spaces, and yields the following quantitative result: *Given* $1 \leqq p_0 < 2$, k, *and* $\varepsilon > 0$, *there exists a* p *with* $p > p_0$ *and an* N, *so that if* R *is a subspace of* L^1 *with* $I_p(R) \geqq N$, *then there exist* $r_1, \cdots, r_k$ *in* R *satisfying* (Δ).

2. It follows from Lemma 6 and the proof of Theorem 8 below that *if* R *is a subspace of* L^1 *and* $1 < p_0 < 2$, *then* $I_{p_0}(R) = \infty$ *if and only if for all* $\varepsilon > 0$ *and positive integers* k, *there are* k *elements* $r_1, \cdots, r_k$ *in* R *satisfying* (Δ).

The next lemma is known but we know of no published proof. For the sake of completeness, we give a (possibly new) argument here.

LEMMA 7. *Let* K *and* η *be given, with* k *a positive integer and* $0 < \eta < 1$. *Then there exists an* $\varepsilon = \varepsilon(k, \eta)$ *so that if* $r_1, \cdots, r_k$ *are elements of* $L^1(\mu)$ *satisfying* (Δ) *for* $p_0 = 1$, *then there exist* k *disjoint measurable subsets* $F_1, \cdots, F_k$ *of* Ω *so that for all* i, $1 \leqq i \leqq k$,

$$\int_{F_i} |r_i(t)|\, d\mu(t) > 1 - \eta\,.$$

Proof. We shall show that ε may be taken equal to $\eta^2/8k^2$. We begin with some elementary considerations. Suppose $0 < \delta < 1$ (δ to be determined later) and $0 \leqq \delta a \leqq b \leqq a$; then $(\delta/2)a \leqq a + b - \sqrt{a^2 + b^2}$. From this, it follows that if $x_1, \cdots, x_k$ are non-negative numbers satisfying $x_i \leqq x_k$ for all i and

$$\delta x_k \leqq \max_{1\leqq i\leqq k-1} x_i\,,$$

then

$$\frac{\delta}{2}\max_i x_i \leqq x_1 + \cdots + x_k - \sqrt{x_1^2 + \cdots + x_k^2}\,. \tag{24}$$

Now let $\phi_1, \cdots, \phi_k$ denote the first k Rademacher functions. (ϕ denotes the function defined on the real line, periodic with period one, such that

$$\phi(x) = 1 \quad \text{if } 0 \leqq x < \frac{1}{2}\,;$$

$$\phi(x) = -1 \text{ if } \frac{1}{2} \leqq x < 1\,;$$

$\phi_j(x) = \phi(2^{j-1}x)$ for all j and x.) We use here only the fact that the ϕ_j's are orthonormal and each of modulus one. Now let $\varepsilon = \eta^2/8k^2$, let $r_1, \cdots, r_k$ satisfy (Δ) for $p_0 = 1$, and let $a_1, \cdots, a_k$ be any k scalars. Then for any real s,

$$\int_\Omega |\sum a_i r_i(t)\phi_i(s)|\, d\mu(t) \geqq (1-\varepsilon)\sum |a_i| .$$

Integrating this inequality with respect to s, changing the order of integration, and using the orthonormality of the ϕ_i's we obtain

$$\begin{aligned}&\int_\Omega (\sum |a_i r_i(t)|^2)^{1/2} d\mu(t)\\ &= \int_\Omega\int_0^1 |\sum a_i r_i(t)\phi_i(s)|\, ds d\mu(t)\\ &\geqq (1-\varepsilon)\sum|a_i| = \int_\Omega \sum |a_i r_i(t)|\, d\mu(t) - \varepsilon\sum|a_i| ,\end{aligned}$$

the last equality holding because $\|r_i\| = 1$ for all i. Hence

$$\int_\Omega [\sum |a_i r_i(t)| - (\sum |a_i r_i(t)|^2)^{1/2}] d\mu(t) \leqq \varepsilon \sum |a_i| . \tag{25}$$

Now for each i, $1 \leqq i \leqq k$, let

$$F_i = \{x\colon \delta\, |r_i(x)| > |r_j(x)| \quad \text{for all } j \neq i,\ 1 \leqq j \leqq k\} .$$

Then the F_i's are disjoint, and moreover for each i and $j \neq i$, $\int_{F_i} |r_j|\, d\mu < \delta$; thus for all j,

$$\int_{\cup_{i\neq j} F_i} |r_j|\, d\mu < (k-1)\delta . \tag{26}$$

Now let $B = \sim\bigcup_{i=1}^k F_i$ and let $t \in B$. Then by (24) and the definition of the F_i's,

$$\frac{\delta}{2}\max_i |r_i(t)| \leqq \sum |r_i(t)| - (\sum |r_i|^2(t))^{1/2} ,$$

hence

$$\int_B \frac{\delta}{2}\max_i |r_i(t)|\, d\mu(t) \leqq \int_B [\sum |r_i(t)| - (\sum |r_i|^2(t))^{1/2}] d\mu(t) \leqq k\varepsilon , \tag{27}$$

the last inequality holding by (25).

Hence by (26) and (27)

$$\int_{\sim F_i} |r_i|\, d\mu < (k-1)\delta + \frac{2k\varepsilon}{\delta} < k\left(\delta + \frac{2\varepsilon}{\delta}\right) \tag{28}$$

for all i. Now let $\delta = \sqrt{2\varepsilon}$; then the definition of ε and (28) yield immediately that $F_1, \cdots, F_k$ satisfy the conclusion of the lemma. Q.E.D.

We are now prepared for the main structural result of this paper.

THEOREM 8. *Let* ν *be a (not necessarily finite) measure,* $1 \leqq p < 2$, *and* R *a closed linear subspace of* $L^p(\nu)$. *Then either*

(i) *there exists a* $p' > p$ *such that* R *imbeds in* $L^{p'}(\nu)$, *or*

(ii) *for all* $\varepsilon > 0$, *there exists a subspace* Y *of* R, *an invertible operator* T *from* Y *onto* l^p, *and a projection* P *from* $L^p(\nu)$ *onto* Y *so that* $\|T\|\,\|T^{-1}\| \leqq 1 + \varepsilon$ *and* $\|P\| \leqq 1 + \varepsilon$.

Proof. We first assume that ν is a probability measure. Suppose $p = 1$ and R is non-reflexive. Then (ii) holds by the results of [7]. On the other hand, if R is reflexive, then by Theorem 5, it suffices to prove that $I_p(R) < \infty$ for some $p > 1$. Since the unit ball of R is weakly compact, it is uniformly absolutely continuous. Thus there exists a $\delta > 0$ so that $f \in R$ and $\|f\| = 1$ implies

$$\int_F |f|\, d\nu < \frac{1}{2} \qquad \text{for all measurable } F \text{ with } \nu(F) < \delta. \tag{29}$$

Now choose k a positive integer so that $1/k < \delta$, and let $\varepsilon = \varepsilon(k, 1/2)$ as defined in Lemma 7. If $I_p(R) = \infty$ for all $p > 1$, then by (the crucial) Lemma 6, we may choose $r_1, \cdots, r_k$ in R satisfying (Δ) for $p_0 = 1$.

Hence by Lemma 7, there are k disjoint measurable sets $F_1, \cdots, F_k$ so that $\int_{F_i} |r_i|\, d\nu > 1/2$ for all i. But for some i, $\nu(F_i) \leqq 1/k < \delta$, contradicting (29).

Now suppose $1 < p < 2$, and that for all M and $\varepsilon > 0$ with $\varepsilon < 1$, there exists on $f \in R$ with $\|f\|_p = 1$ and

$$\int_{\{x: |f(x)| \geqq M\}} |f|^p(x)\, d\nu(x) \geqq 1 - \varepsilon\,.$$

Then the results of [7] imply that (ii) holds. On the other hand, if there is an M and $\varepsilon > 0$ with $\varepsilon < 1$, so that

$$\int_{\{x: |f(x)| \geqq M\}} |f|^p(x)\, d\nu(x) \leqq 1 - \varepsilon$$

if $f \in R$ and $\|f\|_p = 1$, then putting $K = M^{p-1}/\varepsilon$, $\|f\|_p \leqq K\|f\|_1$ for all $f \in R$. Then if X denotes the Banach space whose elements are in R, endowed with the L^1-norm, $d(R, X) \leqq K$ and moreover $I_p(X) \leqq K$ by Theorem 1. It now suffices in view of Theorem 5, to prove that $I_{p'}(X) < \infty$ for some $p' > p$. Suppose this were false. Then for all positive k, we could choose $r_1, \cdots, r_k$ in X, satisfying (Δ) for "ε" $= 1/2$ (with "p_0" $= p$). Let $f_1, \cdots, f_k$ satisfy (7) for $n = k$ and any scalars $c_1, \cdots, c_k$ and let $c_1, \cdots, c_k$ be given scalars. Then

$$\begin{aligned} &\int_\Omega \left(\sum |c_i r_i(w)|^p\right)^{1/p} d\nu(w) \\ (30) \qquad &= \int_0^1\int_\Omega \left|\sum c_i f_i(t) r_i(w)\right| d\nu(w)dt && \text{by (7) and Fubini's theorem} \\ &\geqq \frac{1}{2}\int \left(\sum |c_i f_i(t)|^p\right)^{1/p} dt && \text{by } (\Delta)\,. \end{aligned}$$

Thus $I_p(X) \geqq (1/2)r_k^p$ (as defined in Lemma 3) for all positive integers k, so by Lemma 3, $I_p(X) = \infty$, a contradiction.

To prove the general case, we observe that if λ is a σ-finite measure, then $L^p(\lambda)$ is isometric to $L^p(\mu)$ for some probability measure μ. Now suppose R contains a subspace Y isomorphic to l^p. Then there is a measurable set S of σ-finite measure such that $Y \subset L^p(\nu \mid S)$.

Using what we've already proved and noting that $L^p(\nu \mid S)$ is the range of a contractive projection on $L^p(\nu)$, it now follows that (ii) holds. Now if for all $\varepsilon > 0$ and sets S of finite measure, there is an $r \in R$ so that $\|r\| = 1$ and $\|r \cdot \chi_S\| < \varepsilon$, then we may choose disjoint measurable sets $S_1, S_2, \cdots$ and elements $r_1, r_2, \cdots$ in R so that for all n, $\|r_n \chi_{S_n}\| \geqq 1 - 1/2^n$ and $\|r_n\| = 1$. Standard arguments yield that the closed linear span of the r_j's is isomorphic to l^p.

Thus, assuming that R contains no subspace isomorphic to l^p, there exists a set S of finite measure so that putting $R' = \{r \cdot \chi_S : r \in R\}$, then R' is isomorphic to R. Of course then R' also contains no subspace isomorphic to l^p, and since $(1/\nu(S))\nu \mid S$ is a probability measure, we again have that R' (and hence R) imbeds into $L^{p'}(\nu \mid S)$ for some $p' > p$. Q.E.D.

Remarks 1. If the second alternative fails for R, then in fact there is a set F of σ-finite measure such that R is contained in $L^p(\nu \mid F)$. For if this were not so, there would exist a non-compact operator from R to l^p. Since R imbeds in $L^{p'}(\nu)$ for some $p' > p$, this is impossible in virtue of the results of the Appendix of [13]. We also note that by Theorem 5 (and also the proof of Theorem 8), the condition (i) may be replaced by the condition (i′): there exists a $p' > p$ such that $I_{p'}(R) < \infty$. (Strictly speaking, we only defined $I_q(R)$ for subspaces R of $L^1(\mu)$ for some μ. However, $I_q(R) = \sup I_q(E)$, the sup taken over all finite-dimensional subspaces of R, and hence since all such isometrically imbed in L^1, $I_q(R)$ is unambiguously defined in general, for all $1 < q$, $p \leqq 2$, $R \subset L^p(\nu)$. In fact, $I_q(R)$ can also be defined intrinsically, using the remarks following Corollary 2.)

2. It follows from the results of [4] and Theorem 8 that if the Banach space X is isometric to a quotient space of a reflexive subspace of $L^1(\mu)$, then

every operator from a $C(S)$-space to X is 2-a.s.. The arguments of [4] in question and those of the present paper do not make use of Grothendieck's inequalities; and yield his result that if the Banach space X is such that both X and X^* imbeds in $L^1(\nu)$ for some ν, then X is isomorphic to a Hilbert space. (It follows easily from the results of [7] that such an X must be reflexive.)

The proof of Theorem 8 (together with the techniques of [11] for the case $p > 1$) shows that a quantitative version of Theorem 8 holds, which yields information concerning finite-dimensional subspaces of L^p. The result asserts that given a sequence of subspaces of L^p $(p < 2)$, then either the entire sequence uniformly imbeds in $L^{p'}$ for some $p' > p$, or its members contain almost-isometric copies of l_n^p which are the range of almost contractive projections. Phrased another way, we have

THEOREM 9. *Let* $1 \leqq p < 2$, *k a positive integer, and* $\varepsilon > 0$ *be given. Then there exists a* $K < \infty$ *and a* $p' > p$, *so that for all subspaces X of* $L^p(\mu)$, *either*

(i) $I_{p'}(X) \leqq K$, *and hence X is K-isomorphic to some subspace of* $L^{p'}(\mu)$; *or*

(ii) *There exists a subspace of X which is* $(1+\varepsilon)$*-isomorphic to* l_k^p *and* $(1+\varepsilon)$*-complemented in* $L^p(\mu)$.

We omit the details of the proof (but see the remarks at the end of § 2). This result (for the case $p = 1$) has consequences for the finite-dimensional $\mathcal{P}_\lambda$ spaces. We recall that the projection constant of a finite-dimensional space X, is the minimum of the numbers λ so that X is λ-complemented in Y for all spaces Y containing X. If the projection constant of X is at most λ, X is called a $\mathcal{P}_\lambda$-space. Our next result shows that if χ is $\mathcal{P}_\lambda$ and X is of high enough dimension, then X contains almost-isometric copies of l_n^∞.

COROLLARY 10. *Given* λ, n, *and* ε, *there is an N so that if X is a* $\mathcal{P}_\lambda$*-space of dimension at least N, there exists a subspace of X which is* $(1+\varepsilon)$*-isomorphic to* l_n^∞.

Proof. It is known (but unfortunately unpublished) that there is an m (depending only on λ and ε) so that if a Banach space contains a 2λ-isomorph of l_m^∞, then it contains a $(1+\varepsilon)$-isomorph of l_n^∞. By Theorem 9, we may choose K and $p > 1$ so that for any subspace R of L^1, either R contains a 2-complemented 2-isomorph of l_m^1 or $I_p(R) \leqq K$; let $1/p+1/q=1$. We may assume that $X \subset C[0,1]$ and that there is a projection P from $C[0,1]$ onto X of norm at most λ. It follows that there is a subspace R of L^1 with $d(R, X^*) \leqq$

λ. We shall show that if N is large enough, then $I_p(R) > K$. We have by Corollary 2, that since $d(R^*, X) \leqq \lambda$,

(31) the q-a.s. norm of P is at most $\|P\| I_p(R)\lambda \leqq \lambda^2 I_p(R)$.

Thus, the q-absolutely summing norm of the identity operator on X is at most $\lambda^2 I_p(R)$. But the latter is at least $N^{1/q}$ by the results of [5]. Combining this observation with (31), we have for $N > K^q\lambda^{2q}$, that $I_p(R) > K$, and consequently X contains a 2λ-isomorph of l^∞_m, whence X contains a $(1 + \varepsilon)$-isomorph of l^∞_n. Q.E.D.

Theorem 8 yields that reflexive quotients of $C(S)$-spaces are isomorphic to quotients of $L^p(\mu)$-spaces (for some $p < \infty$), and also incidentally gives a new proof of Grothendieck's theorem that complemented reflexive subspaces of $C(S)$-spaces are finite-dimensional.

COROLLARY 11. *Let X be reflexive and a continuous linear image of some $C(S)$-space. Then there is a $2 \leqq q < \infty$ such that every operator from a $C(S)$-space to X is q-a.s. In particular, there is a probability measure μ so that X is isomorphic to a quotient space of $L^q(\mu)$.*

Proof. X^* is isomorphic to a subspace of $C(S)^*$, which in turn may be identified with $L^1(\nu)$ for some (not necessarily finite) measure ν. But the proof of Theorem 8 easily yields that X^* is in fact isomorphic to a subspace of $L^1(\mu)$ for some probability measure μ, and the conclusion of Corollary 11 now follows from Theorems 5 and 8. (This also shows that if X is a complemented subspace of some $C(S)$-space, then X is finite-dimensional, for then the identity operator on X is q-a.s.) Q.E.D.

Remarks. Of course a quantitative version of Corollary 11 follows from Theorem 9. The version: *Given ε and n, there is a K and a $2 \leqq q < \infty$ so that if X is isometric to a quotient space of $C[0, 1]$, then either X contains a subspace Y with $d(X, l^\infty_n) \leqq 1 + \varepsilon$, or there is a quotient space Y of L^q so that $d(X, Y) \leqq K$.* This result thus implies a finite-dimensional analogue of the result of Pelczynski's that if X is non-reflexive and a quotient of a $C(S)$-space, then X contains an isomorph of c_0 [10].

It is known that every non-reflexive subspace of L^1 contains a subspace isomorphic to l^1 [7], and also that L^p has an unconditional basis for $1 < p < \infty$. Thus in view of the results of [1], we have immediately

COROLLARY 12. *Every closed linear subspace of L^1 of infinite dimension, contains a subspace with an unconditional basis.*

Our final result yields information about subspaces of L^p, analogous to that yielded by Theorem 5 concerning subspaces of L^1. Its proof is an immediate consequence of our previous work, and shall be omitted. (For the implication $3 \Rightarrow 1$ in its statement, see the remark, page 211 of [13].)

THEOREM 13. *Let* $1 < p < 2$, *and let* X *be a closed infinite dimensional subspace of* $L^p(\mu)$. *Then the following are all equivalent*:

1. $I_p(X) < \infty$.
2. *There is a constant* K *so that* $\|x\|_p \leqq K \|x\|_1$ *for all* $x \in X$.
3. X *contains no subspace isomorphic to* l^p.
4. *If* (X_n) *is any sequence of subspaces of* X *with* $\dim X_n = n$ *for all* n, *then* $d(X_n, l_n^p) \longrightarrow \infty$.
5. X *imbeds in* $L^{p'}(\mu)$ *for some* $p' > p$.
6. *There exists a* $p' > p$, *an everywhere positive integrable function* ϕ *with* $\int \phi d\mu = 1$, *and a* $K < \infty$ *so that for all* $x \in X$,

$$\frac{1}{K}\|x\| \leqq \left(\int |x|^{p'}(t)\phi^{1-p'}(t)d\mu(t)\right)^{1/p'} \leqq K\|x\| .$$

7. *Every operator from* X *to* l^p *is compact.*
8. *Every operator from every* $C(S)$*-space to* X^* *is* q*-a.s., where* $1/p + 1/q = 1$.
9. *There exists a surjective* q*-a.s. operator from some* $C(S)$*-space onto* X^*.

Moreover if any of these conditions failed, then for all $\varepsilon > 0$, X *contains a* $(1+\varepsilon)$*-isomorph of* l^p *which is* $(1+\varepsilon)$*-complemented* $L^p(\mu)$.

Remarks. Our observations following Lemma 3 show that $3 \Rightarrow 4$ may be sharpened as follows:

If $X \subset L^p$ is such that there exists a sequence (X_n) of subspaces of X satisfying

$$\frac{d(X_n, l_n^p)}{(\log n)^{1/p}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad \text{then } I_p(X) = \infty ,$$

and consequently X contains a (complemented) isomorph of l^p.

We wish finally, to make some concluding remarks concerning localizations of our theorems. Fix $1 \leqq p < 2$.

Let us say that a Banach space X satisfies $P_{k,\varepsilon}$, if there are elements $r_1, \cdots, r_k$ in X, satisfying (Δ) for $p_0 = p$. Now our results yield that if X is a subspace of L^p, so that for some $0 < \varepsilon < 1$, X satisfies $P_{k,\varepsilon}$ for all k, the X contains an isomorph of l^p. In view of the techniques of [11], this result may be localized as follows:

Given $1 < p < 2$, $0 < \varepsilon < 1$, $0 < \eta < 1$, *and* N, *there exists a* k; *so that if* X *is a subspace of* L^p *satisfying* $P_{k,\varepsilon}$; *then there exists a subspace* Y

of X, *with* $d(Y, l_N^p) \leqq 1 + \eta$, *and a projection* P *from* L^p *onto* Y *with* $\|P\| \leqq 1 + \eta$.

To see this, we let $q = p/(p-1)$ and define (as in [11])

$$C_X = \inf_\phi \sup_{f \in E} \|f\|_p / \|f \cdot \phi^{1/q}\|_1 ,$$

the infimum taken over all non-negative ϕ in L^1 with $\|\phi\|_1 = 1$ and $[0, 1] = \{x: \phi(x) \neq 0\}$. Then the techniques of [11] show that there exists a K (depending only on p, N, and η), so that if $C_X \geqq K$, then X contains a Y with the desired properties. Now for any $X \subset L^p$, $I_p(X) \leqq C_X$ (by the remark following the proof of Corollary 2). Moreover, if X satisfies $P_{k,\varepsilon}$, then our proof of Theorem 8 (specifically, the equalities and inequalities (30)) yields easily that $I_p(X) \geqq (1 - \varepsilon) r_k^p$ (where r_k^p is defined in Lemma 3); and consequently $C_X \geqq (1 - \varepsilon) r_k^p$. The desired result now follows in view of the fact that $r_k^p \to \infty$ as $k \to \infty$. (This argument also shows that there exists a function g_p from the nonnegative extended reals to themselves, with

$$I_p(X) \leqq C_X \leqq g_p(I_p(X)) \qquad \text{for all } X \subset L^p .$$

We don't know if g_p can be taken to be linear, or even to be the identity function.)

Appendix. $p - r$ absolutely summing operators defined on $C(S)$-spaces

Definition. Given Banach spaces X and Y, $1 \leqq p$, $r < \infty$, and an operator $T: X \to Y$; T is called $p - r$ absolutely summing (notation — $p - r$-a.s.) if there is a constant $K < \infty$, so that for any integer n and n elements $x_1, \cdots, x_n$ in X,

$$\left(\sum_{i=1}^n \|Tx_i\|^p\right)^{1/p} \leqq K \sup \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^r\right)^{1/r} , \tag{32}$$

the supremum taken over all $x^* \in X^*$ with $\|x^*\| \leqq 1$. The smallest possible K satisfying (32) for all n, $x_1, \cdots, x_n \in X$, will be called the $p - r$-a.s. norm of T.

An analysis very similar to that of § 1 yields the following

PROPOSITION A1. *Let* $1 \leqq p$, $r < \infty$, $1/p + 1/q = 1 = 1/r + 1/s$, X *a Banach space, and* $K < \infty$. *Then the following statements are all equivalent*:

1. *For every compact Hausdorff space* S *and operator* $T: C(S) \to X$, T *is* $p - r$*-a.s. with* $p - r$*-a.s. norm less than or equal to* $\|T\| K$.

2. *For all* n, $x_1, \cdots, x_n$ *in* X, *and matrices* (y_{ij}) $(1 \leqq i \leqq n,\ 1 \leqq j \leqq n)$ *of scalars*;

$$\left(\sum_i \left\|\sum_j y_{ij} x_j\right\|^p\right)^{1/p} \leqq K \sup_j \left(\sum_i |y_{ij}|^r\right)^{1/r} \sup_{y \in S_{X^*}} \sum_i |\langle y, x_i\rangle| .$$

3. *For all* n, $x_1, \cdots, x_n$ *in* X, *and* $y_1, \cdots, y_n$ *in* X^* ,

$$\sum_j \left(\sum_i |\langle y_i, x_j\rangle|^s\right)^{1/s} \leqq K\left(\sum_i \|y_i\|^q\right)^{1/q} \sup_{y\in S_{X^*}} \sum_j |\langle y, x_j\rangle| .$$

Now if $2 < p < \infty$, it is a theorem of Kwapien [8] that every operator from a $C(S)$-space to L^p is p'-a.s. for all $p' > p$. (This result also follows immediately from our Theorem 5, see the remarks following its proof.) Moreover, by the results of [8] and [15] (see also our Theorem 5) there are operators from a $C(S)$-space to L^p which are not p-a.s. Nevertheless, we do obtain the

PROPOSITION A2. *Let r, p be given with $2 \leqq r < p < \infty$. Every operator from a $C(S)$-space to L^p is $p - r$-a.s.*

Proof. Define s and q by $1/r + 1/s = 1/p + 1/q = 1$. Let $f_1, f_2, \cdots$ be a sequence of independent symmetric identically distributed random variables, stable of exponent s, normalized so that for all scalars $c_1, \cdots, c_n$,

$$\left\|\sum c_i f_i\right\|_1 = \left(\sum |c_i|^s\right)^{1/s} . \tag{33}$$

(In the case of complex scalars, let the f_j's be chosen as in the remark following Theorem 1.) Then we also know, since $1 < q < s \leqq 2$, that there is a constant K (depending only on r and p) so that for all scalars $c_1, \cdots, c_n$,

$$\left\|\sum c_i f_i\right\|_q = K\left(\sum |c_i|^s\right)^{1/s} . \tag{34}$$

We now apply the third statement of Proposition A1. By homogeneity, we may assume that $x_1, \cdots, x_n$ in L^p are given so that

$$\sum_j |\langle y, x_j\rangle| \leqq \|y\|_q \qquad \text{for all } y \in L^q . \tag{35}$$

Now letting $y_1, \cdots, y_n$ be given in L^q, we have that

$$\begin{aligned}
\sum_j \left(\sum_i |\langle y_i, x_j\rangle|^s\right)^{1/s} &= \sum_j \int \left|\sum_i \langle y_i, x_j\rangle f_i(t)\right| dt && \text{by (33)} \\
&= \int \sum_j \left|\sum_i \langle f_i(t) y_i, x_j\rangle\right| dt \\
&\leqq \int \left\|\sum_i f_i(t) y_i\right\|_q dt && \text{by (35)} \\
&\leqq \left(\iint \left|\sum f_i(t) y_i(w)\right|^q dt\, dw\right)^{1/q} && \text{by Hölder's inequality and Fubini's theorem} \\
&= K\left(\int \left(\sum |y_i(w)|^s\right)^{q/s} dw\right)^{1/q} && \text{by (34)} \\
&\leqq K\left(\int \sum |y_i(w)|^q dw\right)^{1/q} && \text{since } \frac{q}{s} < 1 \\
&= K\left(\sum \|y_i\|^q\right)^{1/q} .
\end{aligned}$$

We thus have by Proposition A1, that every operator T from a $C(S)$-space to L^p is $p - r$-a.s. with $p - r$-a.s. norm at most $\|T\| K$. Q.E.D.

Remark. It was previously known that every operator from a $C(S)$-space to L^p is $p-2$ a.s. (c.f. [9]). We also note that in the case of the real scalars, it can be shown that there is a constant C_p depending only on p, so that

$$K \leqq C_p\left(\frac{1}{s-q}\right)^{1/q},$$

where K, p, s, and q are as in the above proof.

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