POINT-WISE COMPACT SUBSETS OF THE FIRST BAIRE CLASS

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Let X be a complete separable metric space. Various characterizations of point-wise compact subsets of the first Baire class of real-valued functions on X are obtained. For example, it is proved that a compact subset is sequentially compact in the topology of point-wise convergence, and moreover (in the case where it is additionally uniformly bounded) that it is compact with respect to the topology induced by the set of Borel probability measures on X. The results are applied to show that l^1 imbeds in a separable Banach space B provided there exists a bounded sequence in B^{**} which has no weak*-convergent subsequence.

§1. Statement of the main result and immediate consequences. Throughout, X denotes a Polish space, i.e. a complete separable metrizable topological space. *We denote by* $\underset{\sim}{B}_1(X)$ *the topological space whose elements are the real-valued Baire-*1 *functions on X, endowed with the topology of point-wise convergence.* A Baire-1 function, or function of the first Baire class, is one which equals a point-wise limit of a sequence of continuous functions on X. The topology of point-wise convergence is of course the product topology on $\underset{\sim}{B}_1(X)$ regarded as a subset of the space of all real-valued functions defined on X.

Our main result is as follows:

MAIN THEOREM. *Let F be a subset of* $\underset{\sim}{B}_1(X)$. *Then the following three statements are equivalent*:

(1) *F is relatively compact.*
(2) *Every countable infinite subset of F has a cluster point in* $\underset{\sim}{B}_1(X)$.
(3) *Every sequence of elements of F has a convergent subsequence. Suppose that F satisfies one and hence all of these conditions.*

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* This has been answered in the affirmative by Charles Stegall: see the addendum at the end.

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Then

(a) *every function in the closure of F is in the closure of a countable subset of F;*

(b) *if F is uniformly bounded (i.e.* $\sup\{|f(x)|: x \in X$ *and* $f \in F\} < \infty$) *and* (f_α) *is a convergent net of elements of F with limit f, then*

(*) $\int f_\alpha \, d\mu \to \int f \, d\mu$ *for all signed Borel measures* μ *on X.*

As we show below the implication (2) $\Rightarrow$ (3) yields that if a separable Banach space B is such that some bounded sequence in B^{**} has no convergent subsequence, then B contains an isomorph of l^1. (An alternate proof of this result has been given by E. Odell (unpublished as of this writing).) Our proof of this implication also yields the main result of [18], namely that if a bounded sequence in a Banach space has no weak-Cauchy subsequence, then it has a subsequence equivalent to the usual l^1-basis. We use results from [15] and also [18]; our arguments here are topological, rather than combinatorial as in [18]. (The known characterizations of Banach spaces containing l^1 are summarized in the third and final section of this paper.)

Let $M(X)$ denote the space of all bounded signed Borel measures on X, and let τ denote the $M(X)$-topology on the bounded members of $\underset{\sim}{B}_1(X)$. That is, a net (f_α) of bounded functions in $\underset{\sim}{B}_1(X)$ τ-converges to a bounded f in $\underset{\sim}{B}_1(X)$ if and only if (*) holds.

Let us define a topological space Y to be *strongly countably compact* if every separable subset of Y has compact closure. It is easily seen that Y is strongly countably compact if and only if for every open cover G of Y and separable subset Z of Y, there is a finite subset F of G with $Z \subset \cup F$. It is also easily seen that the following two conditions are equivalent for a compact Hausdorff space K. (i) Every strongly countably compact subset of K is compact. (ii) Every point in the closure of a subset E of K is in the closure of a countable subset of E.

Now let F satisfy the hypotheses of our Main Theorem. It follows immediately that *F is compact if and only if F is strongly countably compact; if F is uniformly bounded, then this is equivalent to F being* τ*-compact; if F is compact, then F is sequentially compact; and of course if F is uniformly bounded in addition, then the* τ *and* $\underset{\sim}{B}_1(X)$*-topologies on F agree*. This result generalizes the classical Helly selection theorem. Indeed, let $X = [0, 1]$ and $\underset{\sim}{H}$ the set of all (not-necessarily-strictly) increasing functions from X into itself. Then $\underset{\sim}{H}$ is a compact subset of $\underset{\sim}{B}_1(X)$; the Helly selection theorem is simply a restatement of the fact that $\underset{\sim}{H}$ is sequentially compact. It is known that $\underset{\sim}{H}$ is separable but not metrizable (see page 164 of [12]); thus $\underset{\sim}{H}$ is not homeomorphic to a weakly compact subset of a Banach space. On the other hand, if X is compact, F is uniformly bounded and the point-wise closure of F is actually contained in $C(X)$, we obtain the result of Grothendieck that F is relatively weakly compact in $C(X)$ (see [6]). (For any X, $C(X)$ denotes the space of bounded real-valued continuous functions on X under the supremum norm). We also note in passing that our Main Theorem generalizes to the complex-scalars case. (All the assertions follow immediately from the real-scalars case except possibly the implication (1) $\Rightarrow$ (a): Suppose F is a subset of complex $\underset{\sim}{B}_1(X)$ satisfying (1) and g belongs to the closure of F; then setting $\tilde{F} = \{|f - g|: f \in F\}$, $\tilde{F}$ is a relatively compact subset of real $-\underset{\sim}{B}_1(X)$ with 0 in its closure. Thus 0 is in the closure of a countable subset of $\tilde{F}$ which implies g is in the closure of a countable subset of F).

Let F be a relatively compact subset of $\underset{\sim}{B}_1(X)$. We do not know the answer to the following questions: If F is sequentially compact, is F compact (i.e. closed)? Is F sequentially dense in its closure? Obviously an affirmative answer to the second question implies an affirmative answer to the first. We give affirmative answers to these questions in some special cases—see Corollary 6 and the final remarks of Section 2. We also don't know which, if any, of the above results hold on $\underset{\sim}{B}(X)$, the space of all Borel-measurable functions on X.[†] Our proofs, like those of [15], make crucial use of the following result of Baire: *The Baire characterization theorem* ([1]; see also pp. 288–289 of [8]: *Let f be a real-valued function defined on a given X. Then f belongs to the first Baire class on X if and only if for every non-empty closed subset M of X, $f|M$ has a point of continuity relative to the topological space M.*

§2. Proof of the Main Theorem. Most of the arguments are in the same spirit as those of [15] and [18]. The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (a), however, use a "new" technique ("new" in the sense of not being essentially used in [15] and [18]; actually the technique was used by Baire in 1889). The goal of the arguments is to obtain a function badly discontinuous in the following sense:

DEFINITION. *Let Y be a topological space and f a real-valued function defined on Y. We say that f satisfies the Discontinuity Criterion provided there is a non-empty subset L of Y and real numbers r, δ with $\delta >$*

[†] These questions as well as those of section 2 have all been resolved in the affirmative; see the remarks added in proof.

0 *so that for every non-empty relatively open subset U of L, there are y and z in U with $f(y) > r + \delta$ and $f(z) < r$.*

Our next result follows immediately from the definitions and the proof of Lemma 3 of [15].

PROPOSITION 1. *Let Y and f be as in the above definition and suppose f satisfies the Discontinuity Criterion. Then there is a closed non-empty subset K of Y such that $f|K$ has no point of continuity relative to the topological space K. Suppose moreover that there is a uniformly bounded family F of continuous real-valued functions on Y so that f is in the point-wise closure of F. Then F contains a sequence equivalent in the supremum norm to the usual l^1-basis.*

Proof. Let L, r, and δ be chosen as in the above definition. Then $K = \bar{L}$ is the desired closed subset. Now suppose only that $f|L$ is in the point-wise closure of $F|L$; that is, for every $\epsilon > 0$ and finite number $l_1, \ldots, l_n$ of elements of L, there is an f' in F with $|f'(l_i) - f(l_i)| < \epsilon$ for all $1 \le i \le n$. Then the proof of Lemma 3 of [15] (which also uses Proposition 4 of [18]) yields immediately that there is a sequence (f_n) of elements of F so that for all n and scalars $c_1, \ldots, c_n$,

$$\sup_{x \in L} \left| \sum_{j=1}^{n} c_j f_j(x) \right| \ge \frac{\delta}{2} \sum_{j=1}^{n} |c_j| .$$

Since F is uniformly bounded, (f_n) is the desired sequence equivalent in the supremum norm to the usual l^1-basis.

We pass now to the implication (2) $\Rightarrow$ (3). Its proof uses only the "only if" part of the Baire characterization theorem, a category result which is a common exercise in most advanced undergraduate analysis texts. However the proof uses a technique and a fact also used by Baire in his proof of the "if" part of his characterization theorem; namely the technique of construction by transfinite induction and the fact that since a Polish space X has a countable base, X has no strictly descending transfinite sequence of closed subsets (i.e. there is no family $\{K_\alpha : \alpha < \omega_1\}$ of closed subsets of X, indexed by the first uncountable ordinal ω_1, with $K_\alpha \subsetneq K_\beta$ for all $\beta < \alpha < \omega_1$).

Suppose 2 holds. Then F must be point-wise bounded. Hence (2) $\Rightarrow$ (3) follows immediately from the following result:

THEOREM 2. *Let (f_n) be a point-wise bounded sequence of real-valued functions on X such that (f_n) has no point-wise convergent subsequence. Then there exists a non-empty subset L of X and a subsequ-*

ence $(f_n{}')$ point-wise convergent on L so that the limit function f satisfies the Discontinuity Criterion. Consequently $(f_n{}')$ has no Baire-1 cluster points in the topology of point-wise convergence.

Proof. The final assertion follows from the Baire-characterization theorem. Indeed, once $(f_n{}')$ is constructed with L and f as above, if g is any point-wise cluster point of $(f_n{}')$, then g must agree with f on L; hence g must fail to have a point of continuity relative to some closed non-empty subset of X, so $g \notin \underset{\sim}{B}_1(X)$ by the characterization theorem. Our argument for the first assertion, however, makes no use of the characterization theorem.

We introduce the following conventions: N denotes the set of positive integers; every subset M of N shall be taken to be infinite unless otherwise specified. By a sequence we mean a collection of objects indexed by a subset of N. If M and M' are subsets of N with $M' \cap \sim M$ finite, we say that M' is almost contained in M, for which we use the notation $M' \subset_a M$. If $M' \subset_a M$, we say that $(f_n)_{n \in M'}$ is a subsequence of $(f_n)_{n \in M}$.

We first require the following lemma, which follows immediately from Lemmas 5 and 6 of [18] in the uniformly-bounded case:

LEMMA 3. *Let $(f_n)_{n \in N}$ be as in Theorem 2. Then there are an $N' \subset N$ and real numbers r and δ with $\delta > 0$ so that for every $M \subset N'$ there is an x in X satisfying*

$$\begin{aligned} &f_m(x) > r + \delta \qquad \textit{for infinitely many } m \textit{ in } M \\ &\text{and} \\ &f_m(x) < r \qquad\quad\ \ \textit{for infinitely many } m \textit{ in } M. \end{aligned} \tag{1}$$

Proof of Lemma 3. Let $(r_1, \delta_1), (r_2, \delta_2), \ldots$ be an enumeration of all pairs of rational numbers (r, δ) with $\delta > 0$, and suppose the conclusion of Lemma 3 is false. We now choose (infinite) sets $N = M_0 \supset M_1 \supset M_2 \supset \ldots$ as follows: suppose $n \geq 1$ and M_{n-1} has been chosen; then choose $M_n \subset M_{n-1}$ so that every $x \in X$ fails to satisfy (1) for $M = M_n$ and $(r, \delta) = (r_n, \delta_n)$. It then follows that if $M \subset_a M_n$, then every $x \in X$ fails (1) for $(r, \delta) = (r_n, \delta_n)$. This completes the definition of the M_n's by induction.

Now choose, by the standard diagonalization argument, a set M so that $M \subset_a M_n$ for all n. Then for every $x \in X$, there is no pair of rational numbers (r, δ) with $\delta > 0$, satisfying (1). But since $(f_n)_{n \in M}$ is point-wise bounded and non-convergent, there must exist an $x \in X$ so that $\underline{\lim}_{m \in M} f_m(x) < \overline{\lim}_{m \in M} f_m(x)$. Now simply choose rational numbers $(r,$

δ) with $\underline{\lim}_{m\in M} f_m(x) < r < r+\delta < \overline{\lim}_{m\in M} f_m(x)$. Then x satisfies (1), a contradiction.

This completes the proof of Lemma 3. The above argument is used in [4] and is similar to the proof of Lemma 6 of [18]. Thus (as pointed out to the author by several readers) it is not necessary to use transfinite induction to obtain Lemma 5 of [18]. Of course the argument also makes no use of the fact that X is a Polish space. We continue our proof of Theorem 2; we now make essential use both of transfinite induction and the fact that X is Polish.

Let N', r, and δ be as in Lemma 3. *For every $M \subset N'$, let $K(M)$ equal the closure of the set of all x in X satisfying* (1). We then have that for every M and M' with $M' \subset_a M \subset_a N'$, $K(M)$ is a closed non-empty subset of X with $K(M') \subset K(M)$. We next assert that there exists an $M \subset N'$ so that

$$K(M') = K(M) \qquad \text{for all} \quad M' \underset{a}{\subset} M. \tag{2}$$

Suppose this were false; then we could construct a family $(M_\alpha)_{\alpha<\omega_1}$ of infinite subsets of N' so that for all $\alpha < \beta < \omega_1$, $M_\beta \subset_a M_\alpha$ and $K(M_\beta) \subset_{\neq} K(M_\alpha)$; the family $(K(M_\alpha))_{\alpha<\omega_1}$ would thus be a strictly descending transfinite sequence of closed sets, and there is no such family. The details: Let $M_0 = N'$; having chosen M_α, simply choose $M_{\alpha+1} \subset_a M_\alpha$ with $K(M_{\alpha+1}) \neq K(M_\alpha)$; if $\gamma < \omega_1$ is a limit ordinal and the M_α's have been constructed with the desired properties for all $\alpha < \gamma$, choose M_γ almost contained in M_α for all $\alpha < \gamma$, by a standard diagonalization argument. Since by our inductive assumption $\alpha < \beta < \gamma$ implies $K(M_\beta) \subset_{\neq} K(M_\alpha)$, we have that $K(M_\gamma) \subset_{\neq} K(M_\alpha)$ for all such α.

Now choose $M \subset N'$ satisfying (2) and put $K = K(M)$. We then have that for every $M' \subset M$ and every non-empty relatively open subset U of K, there exist $M'' \subset M'$ and y and z in U so that

$$\lim_{n\in M''} f_n(y) \geq r + \delta$$

and (3)

$$\lim_{n\in M''} f_n(z) \leq r.$$

To see this, let M' and U be given. By the definition of $K(M')$ and (2), there exists a y in U so that $f_n(y) > r + \delta$ for infinitely many n in M'. Now choose a subset $\bar{M}$ of M' so that $(f_n(y))_{n\in\bar{M}}$ converges; again, there exists a z

in U so that $f_n(z) < r$ for infinitely many $n \in \bar{M}$; finally, choose $M'' \subset \bar{M}$ so that $(f_n(z))_{n \in M''}$ converges.

Finally, let $U_1, U_2, \ldots$ be a sequence of non-empty relatively open subsets of K, forming a base for the topology of K. We may then choose sequences (M_n), (y_n), and (z_n) so that for all n, y_n and z_n are in U_n, $M_{n+1} \subset M_n$, M_n is an (infinite) subset of N, and (3) holds for $y = y_n$, $z = z_n$, and $M'' = M_n$. Now choose $Q \subset N$ so that $Q \subset_a M_n$ for all n; put $L = \{y_n, z_n : n = 1, 2, \ldots\}$ and define f on L by $f(x) = \lim_{n \in Q} f_n(x)$ for all $x \in L$. Then L is a dense subset of K; if U is a non-empty relatively open subset of L, there exists a V relatively open in K with $V \cap L = U$; hence there is an i with $U_i \subset V$; but then $f(y_i) \geq r + \delta$ and $f(z_i) \leq r$, and y_i and z_i belong to $U_i \cap L \subset U$. Consequently $(f_n') = (f_n)_{n \in Q}$, L, and f satisfy the conclusion of Theorem 2.

Q.E.D

We pass now to the proof of the implication (2) $\Rightarrow$ (1) of the Main Theorem. Suppose (2) holds yet (1) does not. Then F is point-wise bounded; hence the point-wise closure of F in $X^{\mathbf{R}}$ is compact by the Tychonoff theorem, where $X^{\mathbf{R}}$ denotes the space of all real-valued functions on X endowed with the product topology. Therefore, there must exist a non-Baire-1 function f in the point-wise closure of F. By the Baire characterization theorem, there exists a closed non-empty subset K of X so that $f|K$ has no points of continuity relative to K. The proof of Lemma 2 of [15] then yields that f satisfies the Discontinuity Criterion. Indeed, Lemma 2 of [15] asserts precisely this fact under the assumption that f is bounded and K is a compact Hausdorff space; however, its proof used only the Baire category theorem, which of course holds on Polish spaces also; the boundedness-assumption was irrelevant. So, choose $L \subset K$, r, and δ as in the definition of the Discontinuity Criterion; then let $U_1, U_2, \ldots$ be a sequence of non-empty relatively open subsets of L forming a base for the topology of L. For each n choose y_n and z_n in U_n with $f(y_n) > r + \delta$ and $f(z_n) < r$. Let $Q = \{y_n, z_n : n = 1, 2, \ldots\}$. Since Q is countable and f is in the point-wise closure of F, we may choose a sequence (f_n) in F so that $f_n(q) \to f(q)$ for all $q \in Q$. Since Q is dense in L, it follows that $f|Q$ satisfies the Discontinuity Criterion in Q. Again, if g is a point-wise cluster point of (f_n), then $g|Q = f|Q$, hence g has no points of continuity on $\bar{Q}$, so (f_n) has no Baire-1 cluster points by the Baire characterization theorem. Thus (2) fails to hold, a contradiction.

Since trivially (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2), this completes the proof of the equivalences of (1) – (3) of the Main Theorem. We now pass to the implication (2) $\Rightarrow$ (a). The main step of this argument is given by

LEMMA 4. *Let S be a point-wise relatively compact subset of $\underline{B}_1(X)$ so that $0 \in \bar{S}$ and $s(x) \geq 0$ for all $s \in S$ and $x \in X$. Then for all $\delta > 0$ there exists a countable subset H of S so that $\inf_{h \in H} h(x) < \delta$ for all $x \in X$.*

Proof of Lemma 4. Let "H" with or without sub or superscripts denote a countable non-empty subset of S. Suppose the conclusion of Lemma 5 is false. We may then choose a $\delta > 0$ so that putting

$$K(H) = \{x \in X: h(x) \geq \delta \quad \text{for all} \quad h \in H\},$$

then $K(H)$ is non-empty for all H. We have that

$$K(H) \supset K(H') \qquad \text{for all} \quad H \subset H'. \tag{4}$$

We may now construct transfinite sequences $(D_\alpha)_{\alpha<\omega_1}$, $((s_n{}^\alpha)_{n \in N})_{\alpha<\omega_1}$, and $(H_\alpha)_{\alpha<\omega_1}$ with the following properties for all $\alpha < \beta < \omega_1$:

(a) $H_\alpha \subset H_\beta$;
(b) D_α is a countable dense subset of K_α, where K_α equals the closure of $K(H_\alpha)$;
(c) $(s_n{}^\alpha)$ is a sequence of elements of S so that $\lim_{n\to\infty} s_n{}^\alpha(x) = 0$ for all $x \in D_\alpha$;
(d) $H_{\alpha+1} = H_\alpha \cup \{s_n{}^\alpha: n = 1, 2, \ldots\}$.

To see that this is possible, let H_0 be arbitrary. Having chosen H_α, choose D_α to satisfy (b), then choose $(s_n{}^\alpha)$ to satisfy (c); this is possible by the usual diagonalization argument and the fact that $0 \in \bar{S}$; finally simple define $H_{\alpha+1}$ by (d). For β a limit ordinal and H_α defined for $\alpha < \beta$, simply put $H_\beta = \cup_{\alpha<\beta} H_\alpha$; the countability of β and all the H_α's insures the countability of H_β.

These objects having been chosen, it follows from (4), (a), and the definition of K_α that for all $\alpha < \beta < \omega_1$, $K_\alpha \supset K_\beta$. Since X has a countable base and the K_α's are closed, there must exist an $\alpha < \omega_1$ so that $K_\alpha = K_{\alpha+1}$. Now let f be any point-wise cluster point of $(s_n{}^\alpha)$. Then f must vanish on D_α by (c). But for every $x \in K(H_{\alpha+1})$, $s_n{}^\alpha(x) \geq \delta$ for all n, hence $f(x) \geq \delta$. D_α and $K(H_{\alpha+1})$ are both dense subsets of K_α. Thus f satisfies the Discontinuity Criterion, so $f \notin \underline{B}_1(X)$ by the Baire characterization theorem.

This completes the proof of Lemma 4. We note in passing that the argument shows that the conclusion of the lemma holds for any point-wise bounded subset S of $X^{\mathbb{R}}$ such that every cluster-point of a countable infinite subset of S belongs to $\underset{\sim}{B}_1(X)$.

Now suppose that F satisfies (2) (and hence all of (1) – (3)) of the Main Theorem. For each positive integer m, define the map φ_m: $\underset{\sim}{B}_1(X) \to \underset{\sim}{B}_1(X^m)$ by $(\varphi_m f)(x_1, \ldots, x_m) = |f(x_1)| + \cdots + |f(x_m)|$ for all $f \in \underset{\sim}{B}_1(X)$ and all $(x_1, \ldots, x_m) \in X^m$. It is easily seen that the range of φ_m does indeed consist of Baire-1 functions on the Polish space X^m. Now let $g \in \bar{F}$. We may and shall assume without loss of generality that $g \equiv 0$. Indeed if not we simply replace F by $\{f - g: f \in F\}$ which is also a relatively compact subset of $\underset{\sim}{B}_1(X)$. Now φ_m is a continuous map and $\varphi_m(0) = 0$. Hence $S_m = \varphi_m(F)$ is a relatively compact subset of $\underset{\sim}{B}_1(X^m)$ with $0 \in \bar{S}_m$. Thus Lemma 4 applies; we may choose H_m a countable subset of F so that

$$\frac{1}{m} > \inf\{(\varphi_m h)(y): h \in H_m\} \qquad \text{for all} \quad y \in X^m.$$

It follows that $0 \in \bar{H}$ where H is the denumerable subset of F defined by $H = \bigcup_{m=1}^{\infty} H_m$. This completes the proof of (2) $\Rightarrow$ (a) of the Main Theorem.

We now pass to the final step of the proof of the Main Theorem, namely the implication (1) $\Rightarrow$ (b). We first need the following easy consequence of a theorem of Choquet (see Lemma 1 of [15] and also pp. 100–105 of [17]).

For a given X, let $Bd - \underset{\sim}{B}_1(X)$ denote the space of all bounded members of $\underset{\sim}{B}_1(X)$, endowed with the topology of point-wise convergence.

LEMMA 5 *Let X be compact and let K denote the unit ball of $M(X)$ endowed with the weak*-topology relative to $C(X)$. Define a map T from the space of bounded Baire-1 functions on X into the space of real-valued functions on K by*

$$(Tf)(\mu) = \int f \, d\mu \tag{5}$$

for all $f \in Bd - \underset{\sim}{B}_1(X)$ and $\mu \in K$. Then the range of T is a closed subset of $Bd - \underset{\sim}{B}_1(K)$.

Proof. It suffices to show that the range of T consists of those elements of $\underset{\sim}{B}_1(K)$ which are bounded, antisymmetric and affine as functions on K. It's obvious that every element in the range has these properties. Now suppose $f \in \underset{\sim}{B}_1(K)$ is bounded, antisymmetric and affine. Then it follows that there exists an element $\tilde{f} \in M(X)^* = C(X)^{**}$ so that

$\tilde{f}|K = f$. By Lemma 1 of [15], there exists a bounded $h \in \underset{\sim}{B}_1(X)$ so that $Th = f$. This proves Lemma 5.

Now suppose that F is uniformly bounded and satisfies (1) of the Main Theorem. We may assume that F is compact in $\underset{\sim}{B}_1(X)$, by simply working with $\bar{F}$ rather than F, if necessary. Suppose first that X is compact. Then letting T and K be as defined in the above lemma, $T(F)$ is a relatively compact subset of $Bd - \underset{\sim}{B}_1(K)$. Indeed, $T(F)$ is sequentially compact since T is sequentially continuous on uniformly bounded sets and F is compact; thus $T(F)$ satisfies (3) of the Main Theorem, so $T(F)$ satisfies (1); of course $T(F)$ is uniformly bounded, so $\overline{T(F)} \subset Bd - \underset{\sim}{B}_1(K)$. But $T(F)$ is a relatively closed subset of R, the range of T; indeed F is a closed subset of $Bd - \underset{\sim}{B}_1(X)$ and T^{-1} is a continuous map from R back to $Bd - \underset{\sim}{B}_1(X)$. Since R is closed in $Bd - \underset{\sim}{B}_1(K)$ it follows that $T(F)$ is closed in $Bd - \underset{\sim}{B}_1(K)$. Since $T(F)$ is relatively compact, $T(F)$ is already compact. Since T^{-1} is a continuous one-one map carrying $T(F)$ back onto F, $T^{-1}|F$ is a homeomorphism. Thus $T|F$ is itself continuous, which proves (b) for compact X. To handle the general case, suppose (f_α) is a convergent net in F with limit f; let $c = \sup_\alpha |(f_\alpha)|$, and let $\mu \in M(X)$. Given $\epsilon > 0$, we may choose a compact subset K of X so that $|\mu|(\sim K) < \epsilon$, where $|\mu|$ denotes the total variation of μ. Now the restriction mapping of $\underset{\sim}{B}_1(X)$ into $\underset{\sim}{B}_1(K)$ is continuous; hence $F|K = \{f|K: f \in K\}$ is a uniformly bounded compact subset of $\underset{\sim}{B}_1(K)$. As we have already shown, the $M(K)$-topology coincides with the K-topology on $F|K$, so $\lim_\alpha \int_K f_\alpha \, d\mu = \int_K f \, d\mu$. Consequently

$$\overline{\lim_\alpha} \left| \int (f_\alpha - f) \, d\mu \right|$$
$$\leq \overline{\lim_\alpha} \int_{\sim K} |f_\alpha - f| d\mu \leq 2c\epsilon.$$

Since ϵ was arbitrary, (b) follows.

Q.E.D.

The proof of the Main Theorem is now complete. Our next result is a relative of the theorem that an element in the weak-closure of a subset F of a Banach space, is in the norm closure of the convex hull of F.

COROLLARY 6: *Let X be compact and F a uniformly bounded subset of $C(X)$ such that every sequence of functions in F has a point-wise convergent subsequence. Let $(f_\alpha)_{\alpha \in D}$ be a point-wise convergent net of functions in F with limit-function f. Then there exists a sequence of convex combinations of elements of F, which converges to f point-wise.*

Proof. F satisfies (3) of the Main Theorem, hence F is a relatively compact subset of $\underline{B}_1(X)$; in particular, $f \in \underline{B}_1(X)$. We identify $C(X)^*$ with $M(X)$; $T(f)$ is thus a member of $C(X)^{**}$ where T is the map defined by (5) (for all $\mu \in M(X)$). By (b) of the Main Theorem, we have that $T(f)$ is in the weak*-closure of $T(F)$. Thus $T(f)$ is trivially in the weak*-closure of $T(G)$, where G denotes the convex hull of F. But $T(f)$ is a weak* limit of a sequence of elements of $T(C(X))$ since $f \in \underline{B}_1(X)$ (c.f. [15]); hence by the Sublemma of [15] (see the remark immediately following its proof), $T(f)$ is a weak* limit of a sequence of elements of $T(G)$, which immediately yields the conclusion of Corollary 6.

Remarks. 1. We don't know the answer to the following question: Let $F \subset \underline{B}_1(X)$. Is the convex hull of F a relatively compact subset of $\underline{B}_1(X)$ if F is? What about the special case where X is compact and F is a uniformly bounded subset of $C(X)$? In other words, does the convex hull of F satisfy the hypotheses of Corollary 6 if F does? By the results of [18], this is equivalent to the following question: Let F be a bounded subset of a Banach space such that the convex hull of F contains a sequence equivalent to the usual l^1-basis. Does F itself contain such a sequence?*

2. Let B be a separable Banach space and F a subset of B such that every sequence in F has a weak-Cauchy subsequence. Then F must be uniformly bounded; Corollary 6 implies that if F is convex, then F is sequentially dense in its weak*-closure in B^{**}. (This result is implicit in [15]). (b) of the Main Theorem yields that if $B = C(X)$ for a compact X, then the weak topology on F coincides with the topology of point-wise convergence. Hence if $g \in C(X)$ is such that g is in the point-wise closure of F, there exists a sequence of convex combinations of elements of F which converge uniformly to g.

If F is actually a relatively weakly compact subset of $C(X)$, then by a result of Grothendieck [6], every point-wise cluster point of F is continuous; the above result was thus known in this setting. Of course there are non-relatively weakly compact subsets of $C(X)$ which satisfy the hypotheses of Corollary 6, e.g. $F = \underline{H} \cap C(X)$ where $X = [0, 1]$ and $\underline{H}$ is the Helly-space.

3. Corollary 6 implies that if F is a relatively compact subset of $\underline{B}_1(X)$, then F is sequentially dense in its closure provided F is a uniformly bounded convex subset of $C(X)$ and X is compact.

4. Let F be a sequentially compact subset of $\underline{B}_1(X)$ such that every function in $\bar{F}$ has at most countably many discontinuities. Then F is compact. (For example, the Helly space $\underline{H}$ has these properties). To see

this, let $g \in \bar{F}$; without loss of generality, assume $g = 0$. The sequential compactness of F implies that for every countable subset D of X there exists an $f \in F$ which vanishes on D. Now let D be a fixed countable dense subset of X, and let F_D denote the set of all f in F which vanish on D. Then F_D is also sequentially compact and $0 \in \bar{F}_D$; if $f \in F_D$ and $f(x) \neq 0$, then x is not a point of continuity of f. Now let $f_1 \in F_D$. Having chosen $f_1, \ldots, f_n$, choose $f_{n+1} \in F_D$ so that f_{n+1} vanishes on $\{x \in X: f_j(x)$ is discontinuous at x for some $1 \leq j \leq n\}$. Now let f be a cluster-point of (f_j). If $f(x) \neq 0$ for some x, then $f_j(x) \neq 0$ for infinitely many j's and hence x must be a point of discontinuity for infinitely many f_j's. This is impossible; hence we have $f_n \to 0$. (Let $\tilde{A}$ denote the set of all limits of sequences of elements of A. The above argument actually shows that if F is a relatively compact subset of $\underset{\sim}{B}_1(X)$ so that every element in $\bar{F}$ has countably many discontinuities, then $\tilde{\bar{F}} = \bar{F}$).

§3. **Characterizations of Banach spaces containing l^1.** We say that a Banach space B imbeds in a Banach space D if B is isomorphic (linearly homeomorphic) to a subspace of D. Let B denote a separable Banach space and X the unit ball of B^* endowed with the weak*-topology. Putting $F = \{f|X: f \in B^{**}$ and $\|f\| \leq 1\}$, we have that F is a point-wise compact family of real-valued functions on X (assuming the real-scalar field for B). Suppose B^{**} has a bounded-sequence with no weak* convergent subsequence. Then F fails (3) of our Main Theorem, hence F is not contained in $\underset{\sim}{B}_1(X)$. Thus l^1 imbeds in B by the Main Theorem of [15] and Lemma 1 of [15]. This deduction passes "through" the Baire-characterization theorem. In reality, a more direct argument is possible, which we now sketch: Let (g_n) be a sequence in B^{**} with $\|g_n\| \leq 1$ for all n such that (g_n) has no weak* convergent subsequence. Letting $f_n = g_n|X$ for all n, then (f_n) is a sequence in F with no point-wise convergent subsequence. By Theorem 2, there is a subsequence (f_n') of (f_n), a non-empty subset L of X, and an f defined on X, so that f satisfies the discontinuity criterion and (f_n') converges point-wise to f on L. Now let G be a bounded subset of B so that

(Δ) f is in the point-wise closure of $\{g|L: g \in G\}$

where we regard B as a subset of B^{**}. For example, we could take G to be the unit ball of B, by Goldstine's theorem. Since the elements of G are continuous on L, we obtain by the argument of Lemma 3 of [15] that G contains a sequence equivalent to the usual l^1-basis. If $g_n \in B$ for all n, then (g_n) is simply a sequence in B with no weak-Cauchy subsequence.

Then $G = \{g_n: n = 1, 2, \ldots\}$ satisfies (Δ), hence we obtain that (g_n) has a subsequence equivalent to the usual l^1-basis. This is the main result of [18].

We now pass to a summary of characterizations of Banach spaces containing l^1. (For a set Γ, $l^1(\Gamma)$ denotes the space of all scalar-valued f defined on Γ with $\|f\| = \Sigma_{\gamma \in \Gamma} |f(\gamma)| < \infty$. Let $C = C([0, 1])$, $L^1 = L^1(\mu)$ where μ denotes Lebesque measure with respect to the Lebesgue-measurable subsets of the unit interval).

THEOREM: *Let B be a real or complex separable Banach space. Then each of the following conditions is equivalent to the condition that l^1 imbed in B.*

1. *There is a bounded sequence in B with no weak-Cauchy subsequence.*
2. *There is a bounded sequence in B^{**} with no weak*-convergent subsequence.*
3. *There is an element of B^{**} which is not a Baire-1 function on X, where X denotes the unit ball of B^* endowed with the weak*-topology.*
4. *There is an element of B^{**} which is not a weak*-limit of a sequence of elements of B.*
5. *The cardinality of B^{**} is greater than that of B.*
6. *There is a bounded weak* strongly-countably-compact subset of B^{**} which is not weak*-compact.*
7. *There is a bounded weak*-closed convex subset of B^* which is not the norm-closed convex hull of the set of its extreme points.*
8. *L^1 imbeds in B^*.*
9. *$l^1(\Gamma)$ imbeds in B^* for some uncountable set Γ.*
10. *C is a continuous linear image of B.*

The characterization 1 is dealt with in [18]; 2–7 follow from the results of the present paper and [15]. To see the characterization 6, suppose 6 holds. Defining F as at the beginning of this section, we have that F contains a strongly countably-compact non-compact set Y which thus fails (a) of the Main Theorem. Hence F is not contained in $\underset{\sim}{B}_1(X)$, so l^1 imbeds in $\underset{\sim}{B}$ by the characterization 3 (which is proved in [15]). On the other hand, if l^1 imbeds in B, then $(l^1)^{**}$ is weak*-isomorphic to a subspace of B^{**} and βN is homeomorphic to a weak*-compact subset of $(l^1)^{**}$, where βN denotes the Stone-Cech compactification of the positive integers N. Now it is known that there exists a transfinite sequence $(K_\alpha)_{\alpha<\omega_1}$ of relatively closed-and-open subsets of $\beta N \sim N$ with the property that $K_\beta \subset_{\neq} K_\alpha$ for all $\alpha < \beta < \omega_1$. (See [5], problem 6S. In fact, one can choose a transfinite sequence $(M_\alpha)_{\alpha<\omega_1}$ of infinite subsets of N with the property that for all $\alpha < \beta < \omega_1$, $M_\alpha \cap \sim M_\beta$ is infinite and $M_\beta \subset_a M_a$. Then setting $K_\alpha = \bar{M}_a \cap \sim M_\alpha$ for all α (where N is regarded as a subset of βN), $(K_\alpha)_{\alpha<\omega_1}$ has the desired property). Then $(\cup_{\alpha<\omega_1} \sim K_\alpha) \cap (\beta N \sim N)$ is a strongly countably-compact non-compact subset of βN. The characterizations 8–10 were demonstrated by Pelczynski in [16] under a special assumption which was later removed by Hagler in [7]. Actually, the fact that 7 holds provided l^1 imbeds in B is a consequence of 10. Indeed, 10 implies that C^* weak*-imbeds in B^*. The purely atomic measures on [0, 1] of norm at most one constitute a norm-closed subset of C^* which contains all the extreme points of the unit ball of C^*; thus assuming 10, the image of this set under the weak*-embedding yields a subset of B^* satisfying the property in 7.

It is known that for separable B, B^* is non-separable if (see [13], [2], and [14]) and only if (see [9] and [19]) there is a bounded norm-closed convex subset of B^* which is not the norm-closed convex hull of its extreme points. See [10] and also [3] for examples of B separable with B^* non-separable, for which l^1 does not imbed in B.

The characterizations 1 and 8 do not require the assumption of the separability of B. Also it is proved in [16] that for general B, if l^1 imbeds in B then C^* imbeds in B^*. Hagler has generalized the characterization 9 by showing in [7] that l^1 imbeds in B (not necessarily separable) provided there is a set Γ so that $l^1[\Gamma]$ imbeds in B^* and $\dim B < \text{card } \Gamma$, where $\dim B$ is the smallest cardinality of a subset of B with dense linear span.

We conclude with some open questions and comments; throughout as above, B denotes a real or complex Banach space.

Q 1. Let B^* have a bounded sequence with no convergent subsequence. Does B contain an isomorph of $l^1(\Gamma)$ for some uncountable set Γ?

The theorem of this section shows that the answer is "yes" provided B is isomorphic to the dual of a separable space Y. Indeed, then the characterization 2 yields that l^1 imbeds in Y, hence in fact $l^1(\mathbf{R})$ imbeds in Y^* by the characterization 10. E. Odell has given a direct proof of this special case (unpublished as of this writing); hence his argument yields an alternate proof of the characterization 2 via the characterization 9.

Q 2. Does B contain an isomorph of l^1 if B satisfies the same hypotheses as in Q 1?

Using the results of [11] and [18], W. B. Johnson has shown that the answer to Q 2 is "yes" provided c_0 is not a continuous linear image of B

(unpublished as of this writing). By the results of [11], these assumptions imply that B is non-separable.

Q 3. Does B contain an isomorph of l^1 provided every infinite-dimensional subspace of B has a non-separable dual?

An affirmative answer to Q 3, in view of Johnson's result and the results of [11], implies an affirmative answer to

Q 4. Does every B have a separable infinite-dimensional quotient-space?

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Addendum. The second question of Remark 1 following Corollary 6 has been affirmatively answered by Charles Stegall. His observations may be combined with yet another characterization of Banach spaces containing l^1, due to E. Odell, as follows: An operator from one Banach space to another is called a Dunford-Pettis (D.P.) operator provided it carries weak-Cauchy sequences to norm-Cauchy sequences. Odell's characterization asserts that a Banach space contains an isomorph of l^1 if and only if it admits a D. P. operator into some Banach space which is not compact. In fact, the observations of Odell and Stegall yield: *A subset S of a Banach space B is weakly pre-compact if and only if for every Banach space X and every D. P. operator* $T: B \to X$, $T(S)$ *is a relatively compact subset of S.* ("S is weakly pre-compact" is defined to mean that every sequence in S has a weak-Cauchy subsequence.) It follows immediately from this characterization that the convex hull of a weakly pre-compact set is also weakly pre-compact. The characterization is proved as follows: it suffices to show the "if" assertion since the "only if" assertion is immediate. Let S be a non-weakly pre-compact set. If S is unbounded, then by the uniform boundedness principle there is continuous linear functional f on B with $f(S)$ unbounded; assume that S is bounded. Now the results of [18] and [4] yield that a bounded set is weakly pre-compact if and only if it does not contain a sequence equivalent to the usual l^1-basis. Thus we may choose a sequence (s_n) in S which is equivalent to the usual l^1-basis. Let (g_n) be a uniformly bounded sequence of measureable functions in the unit interval, so that $\{g_n: n = 1, 2, \ldots\}$ is not relatively compact in the L^1-topology (for example one could let the g_n's be the Rademacher functions). Let $T: [s_n] \to L^\infty$ be the operator defined by $T(\Sigma\, a_j s_j) = \Sigma\, a_j b_j$ for all choices of scalars $a_1, a_2, \ldots$ with $\Sigma |a_j| < \infty$, (where $[s_n]$ denotes the closed linear span of the s_n's). T is of course a bounded linear operator. Since L^∞ is an injective Banach space, there exists a bounded linear operator $\tilde{T}: B \to L^\infty$ with $\tilde{T}|[s_n] = T$. Let $i: L^\infty \to L^1$ be the canonical injection. Since i is weakly compact, i is a D.P. operator (see [6]), hence so is $i\tilde{T}$. But $i\tilde{T}$ $(\{s_n: n = 1, 2, \ldots\}) = \{g_n: n = 1, 2, \ldots\}$ is a non-relatively compact subset of L^1. It is worth pointing out that since i is an absolutely summing operator, $i\tilde{T}$ is also; that is, whenever (b_j) is a sequence in B such that $\Sigma |b^*(b_j)| < \infty$ for all b^* in B^*, $\Sigma \| i\tilde{T}(b_j) \| < \infty$.

If we let the g_n's be the Rademacher functions, we have that $i\tilde{T}$ is actually a non-compact D.P. operator with range in L^2. It is in fact possible to construct a D.P. operator from B onto L^2, so that the image of the e_n's is a dense subset of the unit ball of l^2; see Ovsepian and Pełczyński, Seminaire Maurey-Schwartz 1973–1974, Exposé $N^{\cdot}$ XX.

Added in Proof: All of the questions raised in sections 1 and 2 have been answered in the affirmative by J. Bourgain, D. H. Fremlin and M. Talagrand in a paper entitled "Pointwise compact sets of Baire measurable functions," That is, if F is a relatively compact subset of $B_1(X)$ for X a Polish space, then F is sequentially dense in its closure and if F is in addition uniformly bounded, its convex hull is also relatively compact. Also the same conclusions hold if F is a countable relatively countably compact subset of $B(X)$ and moreover F is then relatively compact. The authors use the Main Theorem as section one and deep topological reasoning. Letting B satisfy the hypotheses of the Theorem of section 3, one obtains the following condition equivalent to 1–10: *there exists a bounded subset of B^{**} which is not weak* sequentially dense in its weak* closure.* A more direct proof of the equivalence of 2 of this theorem with the existence of an isomorph of l' in B appears in the author's paper, "Point-wise compact subsets of the first Baire class, with some applications to Banach space theory," Aarhus Universitet Various Publications Series No. 24, Denmark 1974, 176–187. The argument of E. Odell mentioned after Q3 of section 3 appears in his MIT Ph.D. thesis, while the result of W. Johnson mentioned after Q2 will appear in a joint paper coauthored with J. Hagler and entitled "On Banach spaces whose dual balls are not weak* sequentially compact."