

1. Express this number in ordinary base-10 decimal notation:

$$(1 \times 16^{-1}) + (2 \times 16^{-2}) + (3 \times 16^{-3})$$

Answer: C.  $291/4096 = 0.07104\dots$  You can simply add the fractions shown or compute a common denominator.

Want more practice? What is the real number written in hex as “0x89.ABCD”? How about the repeating (hexa)decimal expansion “0x0.CAFECAFECAFE...” ?

2. How big is the set of fake words that use only the letters o and i?

Answer: B. countably infinite (i.e. as large as the set of natural numbers) You can list all the one-letter words first (o, i) then all the two-letter words (oo,oi,io,ii), then the threes, the fours, etc.

Equivalently just start counting in binary; remove the leading “1” and then change all 0’s to o’s and all 1’s to i’s:

#1 :	1	→	(no word)
#2 :	10	→	<i>o</i>
#3 :	11	→	<i>i</i>
#4 :	100	→	<i>oo</i>
#5 :	101	→	<i>oi</i>
#6 :	110	→	<i>io</i>
#7 :	111	→	<i>ii</i>
#8 :	1000	→	<i>ooo</i>

and so on.

3. The molding that I bought was one inch too long. About how high off the floor is this end of the molding?

Answer: D. 17 inches. Use the Pythagorean Theorem on the right triangle formed on the wall by the molding and the floor:  $145^2 = 144^2 + h^2$ , where  $h$  is this height; this means  $h^2 = 289$  so  $h = 17$ .

I’ll bet you didn’t think one extra inch of molding would create such a bad fit!

4. The notation “0xB4” is hexadecimal notation for a certain integer. Express the same integer in binary notation.

Answer: It’s 10110100 in binary (i.e 180, in decimal).

This is actually easier than it looks: since  $16 = 2^4$ , we can simply translate each “digit” of hexadecimal into four bits in binary. That is, 0xB (eleven) is 1011 in binary and of course 0x4 (four) is 0100, so

$$0xB4 = (11 \times 16) + (4 \times 1) = (2^3 + 2^1 + 2^0) \times 2^4 + (2^2) \times 2^0 = (2^7 + 2^5 + 2^4) + (2^2)$$

This ”trick” only works since the base used in one notation (16) is a power of the base used in the other (2).

5. Using whatever technique you prefer, compute the difference  $X - Y$  where  $X$  and  $Y$  are the integers written in base-3 notation as  $X = 12121$  and  $Y = 1201$ .

Answer: These are  $X = 151$  and  $Y = 46$  (when written in base-ten) so the difference is 105; that number is expressed in base-3 as “10220”.

You can also compute this directly in base-3, where the extra numbers in the top line are the scribbles you make when “borrowing” in a subtraction problem:

$$\begin{array}{r}
 \phantom{-} \phantom{1} \phantom{2} \phantom{1} \phantom{2} \phantom{1} \\
 \phantom{-} \phantom{1} \phantom{2} \phantom{1} \phantom{2} \phantom{1} \\
 - \phantom{1} \phantom{2} \phantom{1} \phantom{2} \phantom{1} \\
 \hline
 \phantom{1} \phantom{2} \phantom{1} \phantom{2} \phantom{1} \\
 \hline
 1 \phantom{2} \phantom{1} \phantom{2} \phantom{1} \phantom{2} \phantom{1}
 \end{array}$$

6. Here are two sets of numbers:

$K$  = the set of all rational numbers between 0 and 1 inclusive

$L$  = the set of all rational numbers between 10 and 20 inclusive

Demonstrate that the sets  $K$  and  $L$  have the same cardinality by describing a one-to-one correspondence between the elements of  $K$  and the element of  $L$ .

Answer: Pair off each element  $x$  in  $K$  with  $y = 10(x + 1)$  in  $L$ . (You should, strictly speaking, observe that this  $y$  is rational whenever  $x$  is, that  $y \in [10, 20]$  whenever  $x \in [0, 1]$ , that every element of  $L$  is such a  $y$  for some  $x \in K$ , and that different  $x$ 's are matched with different  $y$ 's. But I'll let you use your intuition to skip all that — this time.)

7. The Acme Marble Company sells some very cool packages of marbles. ... Tell me how to make such a list; also please clearly spell out what the first four items on your list are, and finally tell me how far down on the list I should expect to see the red marble labeled “100”.

Answer: I might rotate through the three colors, listing the balls of each color in sequence. In other words my list goes

- #1 : blue 1
- #2 : red 1
- #3 : green 1
- #4 : blue 2
- #5 : red 2
- #6 : green 2
- ...

The red 100 will be the 299th entry on this list, if I counted right, and similarly I think it's clear that the list mentions every marble of every color.

8. If  $R$  is the set of all right triangles, and  $S$  is the set of all isosceles triangles (triangles having two sides of equal length) what is the intersection of  $R$  and  $S$ ?

Answer: Observe that we're being asked to intersect SETS (of triangles); if you tried to intersect two (or more) triangles, you're not reading the question right.

The intersection is the set of right isosceles triangles, a.k.a. the "45-45-90" triangles.

9. What's the shortest path the bug can take to get home?

Answer: A simple path travels 5 inches to the edge and then 7 inches more from the edge to the house. But that's a foot-long journey. It's shorter to go up an inch to the top, then go diagonally for  $\sqrt{5^2 + 7^2} = 8.60$  inches, then go down another inch: only 10.6 inches!

If you rip the box open where the left and front faces meet, you can lay these three edges flat: put the top face in the middle, the front face attached below and the left face attached to the left. The three-step path can be drawn on this flat diagram but immediately you see it isn't the shortest path — you can draw a straight line on this diagram and that will be shorter; in fact its length will be  $\sqrt{(5+1)^2 + (7+1)^2} = 10$  inches exactly.

Bug geometry is not always intuitive. For example, the point on the box that's furthest away (for the crawling bug) from one corner is *not* necessarily the opposite corner!

10. How long is the longest stick that I could fit inside the cube?

How about inside the tetrahedron?

How about the (regular) octahedron?

Answer: In all three cases the endpoints of the stick would be a pair of points on the shape that are furthest apart. (That statement is true precisely because these shapes are *convex*, which we discussed briefly in class). I think it's pretty clear that you can't get two points further apart than by choosing two vertices.

Well, on the tetrahedron, any two vertices are one foot apart, so that's the longest stick that'll fit. Not a very useful packaging system!

On the octahedron every vertex is one foot away from all the other vertices *except* the vertex that's directly opposite it. Those two vertices can be thought of as endpoints of a diagonal on the base of the two pyramids that glue together to make the octahedron. Well, the diagonal of a 1-foot square has length  $\sqrt{2}$ .

On the cube there are some pairs of vertices that are one foot apart, and others that are  $\sqrt{2}$  feet apart, but also the ends of any diagonal (e.g. the points with Cartesian coordinates  $(0,0,0)$  and  $(1,1,1)$ ) are a distance of  $\sqrt{3}$  apart. Remember that the next time you want to mail a 20-inch rod and the Post Office tells you that you can't ship a box that is any longer than one foot across in any direction!