M361K – Spring 2022 – Rusin – HW2 – due 11:59pm Feb 9

1. The axioms for the real numbers don't assume an operation called "division". Instead, we use the notation "a/b" to mean "the product of a and the (unique) multiplicative inverse of b", i.e. $a \cdot b^{-1}$.

Prove that for all nonzero real numbers a, b, c, d we have

$$\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{a \cdot c}{b \cdot d}$$

2. Prove that if $a, b, c, d > 0$ then $(a + c)/(b + d)$ lies between a/b and c/d.

3. Prove that if $f : \mathbf{R} \to \mathbf{R}$ is a function for which the following are true

$$\text{for every } x, y \in \mathbf{R},\ f(x + y) = f(x) + f(y)$$

$$\text{for every } x, y \in \mathbf{R},\ f(x \cdot y) = f(x) \cdot f(y)$$

then $f(x) = x$ for every rational number x.

(Hint: First prove that $f(x) = x$ for $x = 0$ and $x = 1$.Then prove it by induction for every $x \in \mathbf{N}$, and then for every $x \in \mathbf{Z}$.)

It is an interesting question to ask whether there can exist such functions f for which there are any real numbers x with $f(x) \neq x$. Remember that all the axioms for $\mathbf{R}$ (other than Completeness) are satisfied by other structures, and for some of these other structures, the answer is "yes". Here's an example. Let X be the set $\mathbf{Q}^2$ of ordered pairs of rational numbers (a, b); define addition by $(a, b) + (a', b') = (a + a', b + b')$ but define multiplication by $(a, b) \times (c, d) = (ac + 2bd, ad + bc)$. (In effect, $(0, 1)$ is "$\sqrt{2}$" and these are the numbers $a + b\sqrt{2}$.) Then the function defined by $f(a, b) = (a, -b)$ satisfies the conditions of problem 1, but is not the identity function.

4. Let $\mathbf{R}[X]$ denote the set of polynomials in one variable X, having real coefficients. Let P be the set of polynomials whose leading coefficient is positive, and assume addition and multiplication are defined on usual. Which axioms for the real numbers does $R[X]$ violate?

(If you prefer, you may use the symbol ∞ instead of X; do you see why this is appropriate?)

You may wish to think about whether it is possible to modify the set $\mathbf{R}[X]$ so that it does satisfy the axioms for the real numbers.

5. Proce that this is a metric space

$$X = \mathbf{Z} \quad \text{and} \quad d(x, y) = \frac{1}{2^n} \quad \text{where} \quad |x - y| = 2^n\, m \quad \text{with } m \text{ odd}$$

6. (Important) Recall that in any metric space X, a subset $U \subseteq X$ is called *open* if it is a union of balls in X. (A ball is a set

$$B_r(a) = \{x \in X \mid d(x, a) < r\}$$

for any real r and any $a \in X$.)

Show that a set U is open if and only if

$$\forall u \in U \exists r > 0 \, (B_r(u) \subseteq U)$$

Extra Credit. Since $\mathbf{Q}$ is a countable set, list the rational numbers in any order:

$$\mathbf{Q} = \{x_1, x_2, x_3, \ldots\}$$

Let $U = \bigcup_{n=1}^{\infty} B_{2^{-n}}(x_n)$, which is an open set.

(a) Show U contains every rational number.

(b) Show U does not contain every real number. (Hint: How big of a subset of the real line could U be?)