M365C (Rusin) HW11 – due Thursday, Dec 5 2019

REMINDER: There will be NO CLASS on Tuesday, Nov 26.

1. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and everywhere-positive, and satisfies the condition $f(a+b) = f(a)f(b)$ for all real a and b. Show that for some constant c we have $f(x) = \exp(cx)$.

2. Compute $\lim_{x\to 0}(1+x)^{1/x}$. For once, I do *not* want you to blindly assume all the calculation tricks you used in Calculus; please be prepared to actually justify your calculations with definitions and theorems used in class or in the text.

3. Just as we defined the logarithm as a certain integral, let us define a function $A : \mathbf{R} \to \mathbf{R}$ by

$$A(x) = \begin{cases} \int_{[0,x]} \frac{1}{1+t^2}\, dt & \text{if } x \geq 0 \\ -\int_{[x,0]} \frac{1}{1+t^2}\, dt & \text{if } x < 0 \end{cases}$$

Prove the following:

(a) $A(0) = 0$ and $A(-x) = A(x)$ for all $x > 0$.

(b) A is continuous everywhere

(c) A is differentiable everywhere

(d) A is increasing everywhere

(e) The values of A are bounded. (So the number $2\sup(A(x))$ is well-defined. By tradition this number is named "π" .)

(f) A has an inverse function $T : (-\pi/2, \pi/2) \to \mathbf{R}$ which is differentiable. Compute the derivative $T'(x)$. (You may express your answer in terms of x and T.)

4. Is there a function $f : \mathbf{R} \to \mathbf{R}$ which has the property that $f \circ f = \exp$? Let us try to construct one.

(a) Suppose a is any number in the interval $(0,1)$, and $f_1 : [0,a] \to [a,1]$ is a continuous, increasing function with $f_1(0) = a$ and $f_1(a) = 1$. Show that there is a unique continuous function $f_2 : [0,1] \to \mathbf{R}$ which has the property that $f_2(x) = f_1(x)$ for all $x \leq a$ and $f_2 \circ f_2 = \exp$ on $[0,a]$. (The first condition — that f_2 agrees with f_1 on the domain of the latter — is expressed by saying that f_2 is a *continuous extension* of f_1.)

(b) Show similarly that f_2 has a continuous extension to a function $f_3 : [0, e^a] \to \mathbf{R}$ which has the feature that $f_3 \circ f_3 = \exp$ on $[0,1]$. Continue by induction to construct a function $f : [0,\infty) \to \mathbf{R}$ which has $f \circ f = \exp$ on all of $[0,\infty)$. (Be sure to explain why your function f is defined on all of $[0,\infty)$.)

(c) Show that if f_1 is differentiable on $(0,a)$ then f_2 is differentiable on both $(0,a)$ and on $(a,1)$. Give a condition that guarantees that f_2 is also differentiable at a.