Perhaps it would be helpful to see worked-out solutions to the homework due Monday *before* Monday's test ...

Problem 1c on p.243 (sect 4.3) asks you to find the extrema of a functional $J(y) = \int_0^1 xyy'\,dx$. A priori this is a textbook example of the Euler-Lagrange equations, with $L(x, y, z) = xyz$. Thus the differential equation to be solved is

$$\frac{\partial L}{\partial y}(x, y, y') = \frac{d}{dx}\left(\frac{\partial L}{\partial z}\right)(x, y, y')$$

or simply $xy' = y + xy'$. The only solution is $y = 0$.

However, we are supposed to be optimizing J in the subset of $C^2[0, 1]$ where $y(0) = 0$ and $y(1) = 1$, and this function is not in here! Thus there is no (local) minimum nor maximum of this functional on this subset.

If the second boundary condition had instead been $y(1) = 0$ then indeed the function $y = 0$ would be the only candidate extreme point. But is it actually an extremum, and if so, is it a (local) min or a local max? We didn't address this in class but we had the tools to do so. Recall we turned this into a one-variable min/max problem by considering the function $j(\varepsilon) = J(f + \varepsilon g)$ where g was any fixed smooth function vanishing at the endpoints. We characterize the extrema f by the condition $j'(0) = 0$ (for each g), but we could go further: in calculus you learn that j attains a local minimum at $\varepsilon = 0$ if $j''(0) > 0$ and similarly for a local maximum. So let us compute the *second* derivative of j as we did the first derivative:

$$\begin{aligned}
j''(\varepsilon) &= \frac{d^2}{d\varepsilon^2}\int_0^1 L(x, f(x) + \varepsilon g(x), f'(x) + \varepsilon g'(x))\,dx \\
&= \int_0^1 \frac{d^2}{d\varepsilon^2} L(x, f(x) + \varepsilon g(x), f'(x) + \varepsilon g'(x))\,dx \\
&= \int_0^1 \frac{\partial^2 L}{\partial y^2} g^2 + 2\frac{\partial^2 L}{\partial y \partial z} gg' + \frac{\partial^2 L}{\partial z^2}(g')^2\,dx,
\end{aligned}$$

the last integrand evaluated as usual at $(x, y, z) = (x, f(x) + \varepsilon g(x), f'(x) + \varepsilon g'(x))$. Of course we are only interested in the sign of $j''(0)$. In our case, $L(x, y, z) = xyz$ so the only part of this that survives is $j''(0) = \int_0^1 2xg(x)g'(x)\,dx = \int_0^1 x\frac{d}{dx}(g(x))^2\,dx$. We may evaluate this integral by parts to get $x(g(x))^2 - \int_0^1 g(x)^2\,dx$; since g must vanish at the endpoints, this definite integral evaluates to $-||g||^2$ and in particular is negative. Thus $j''(0) < 0$ giving j a local maximum at $\varepsilon = 0$. Since this is true for every g, we see the function $y = 0$ is a local maximum for the functional J.

More generally, $-2J(y)$ comes out to $||y||^2 - (b\,y(b)^2 - a\,y(a)^2)$ for any function $y \in C^2[a, b]$. So this J is not just a made-up functional of no import: asking for an extremum of J on a set of functions passing through two specified points is simply asking for the largest (or smallest) function y (in the L^2 sense). Perhaps you can see from staring at the graphs that among all the functions whose graphs connect $(0, 0)$ to $(1, 1)$, there are some whose size is arbitrarily small (they enclose only tiny areas under their graphs) but no smooth function can actually enclose zero area. Thus there is no single smallest function, and so we should not be surprised that problem 1c does not lead us to any solution!

Problem 7 on p.243 (sect 4.3) is similar, with $L(x, y, z) = (1+x)z^2$. The E-L equation now is $0 = (d/dx)((1 + x)2(y'))$, whose solutions obviously require $(1 + x)(y') = C$ for some constant C. The solutions then satisfy $\frac{dy}{dx} = \frac{C}{1+x}$ which in turn requires $y(x) = C \log|1+x| + C'$. From $y(0) = 0$ we deduce $C' = 0$ so $y = C \log(1 + x)$. When the second boundary condition is $y(1) = 1$ we compute $C = 1/\log(2)$ and then have our unique solution (actually a local minimum).

The other alternative boundary condition yields no solution. Indeed, consider the prior equation specifying dy/dx; that ODE implies that y' will not be zero for any x — unless it's identically zero, meaning y is constant.

Problem 5a on p.253 (sect 4.4) is a free-boundary problem. The E-L equation to be satisfied is $2y = (d/dx)(2y')$, i.e. $y'' = y$ and as you know the solutions are the functions $y = Ae^x + Be^{-x}$ for any constants A, B. The boundary condition $y(0) = 1$ then gives $A+B = 1$. The natural-boundary condition here is $2y'(1) = 0$, giving a second constraint, $Ae - Be^{-1} = 0$. Solve these two equations for A, B to deduce

$$y = \frac{e^{(x-1)} + e^{-(x-1)}}{e - e^{-1}}.$$

Note that this J measures $||y||^2 + ||y'||^2$, so minimizing J is an attempt to keep both y and y' small. So you should not be too surprised that the natural-boundary condition ends up insisting that y be flat at the right edge. Have a look at the graph of our solution and see if you agree that it does manage to keep $||y||^2 + ||y'||^2$ smaller than any other curve that starts at $(0, 1)$!

Problem 1 on p.271 (sect 4.6) is a constrained optimization problem. We wish to find a function y and a real number λ that optimize $H = J + \lambda K$ where K is the functional whose vanishing specifies our constraint, in our case $K(y) = \int_0^\pi y^2\, dx - 1 = \int_0^\pi y^2 - 1/\pi\, dx$. For each λ, the best y is the one that optimizes $\int_0^1 (y'^2 + \lambda(y^2 - 1/\pi))\, dx$, and very much as in the previous problem, the E-L equation specifies that this y must satisfy $y'' = \lambda y$. Depending on the sign of λ, that function must have one of the forms $Ae^{\mu x} + Be^{-\mu x}$, $A + Bx$, or $A \sin(\mu x) + B \cos(\mu x)$ for some A and B, where $\mu = \sqrt{|\lambda|}$. In each case we determine A, B, μ from the three equations $y(0) = 0, y(\pi) = 0, \int_0^1 y^2\, dx = 1$. The first two cases lead to the conclusion $A = B = 0$ so y is the zero function, which does not meet the third constraint. In the other case, though, the three conditions lead respectively to: $B = 0$, $\mu \in \mathbf{Z}$, and $A = \sqrt{2/\pi}$. So in this case we have found multiple local extrema: $y = \sqrt{2/\pi} \sin(nx)$ for $n = 1, 2, \ldots$. In particular, among all functions of a given size (in the L^2 sense), the sine function is the smoothest (in the sense of minimizing $||y'||$).

Problem 6 on p.271 generalizes the previous one. Now the Euler-Lagrange equation is $(q + \lambda r)(2y) = (d/dx)(2py')$, which is to say $-(py')' + qy = \lambda ry$, where the parameter λ is a priori free, but as in the previous problem the boundary conditions and the side constraint $\int_0^1 ry^2\, dx = 1$ constrain λ to a countable, discrete set (see section 5.2).