1. (20 pts.) Compute the following limits

(i) \( \lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^{3n} \)

(ii) \( \lim_{x \to 0} x^{-1} \int_{3}^{3+x} \cos(\pi y^2) \, dy \)

(iii) \( \lim_{n \to \infty} \sum_{k=0}^{n} \frac{3^k}{k!} \)

(iv) \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k\pi}{n^2} \sin\left(\frac{k\pi}{n}\right) \)

(v) \( \lim_{x \to \infty} x \left(1 - e^{-(1/x)}\right) \)

**ANSWER:**

(i) Exponentiation of real numbers \( a^b \) (with \( a > 0 \)) may be written as \( e^{b \ln(a)} \); since the exponential function is continuous we can then compute \( \lim a^b \) as \( e^{\lim(b \ln a)} \). In our case this requires that we compute \( \lim_{n \to \infty} 3n \ln(1 - (2/n)) \). We may substitute \( n = 1/u \); then we need the limit as \( u \to 0^+ \) of \( 3 \ln(1 - 2u)/u \). With one application of L'Hôpital's Rule this limit is seen to be \(-6\). So the original limit evaluates to \( e^{-6} \).

(ii) Writing this as \( \lim_{x \to 0} \frac{F(x)}{x} \) we see that this again may be computed using L'Hôpital's Rule (since clearly the integral \( F(x) \) will vanish when \( x = 0 \)). But \( F'(x) = \cos(\pi(3 + x)^2) \) by the Fundamental Theorem of Calculus, so the limit involved in L'Hôpital's Rule is simply \( \cos(9\pi) = -1 \).

(iii) This is the limit of the partial sums of an infinite series \( \sum_{k \geq 0} 3^k/k! \). But we recognize this as the Taylor series of the exponential function, evaluated at \( x = 3 \). Hence the value of this limit is \( e^3 \).

(iv) This may be written \( \lim_{n \to \infty} \sum_{k=1}^{n} F(x_k) \Delta x \), where \( F(x) = \frac{1}{\pi} x \sin(x) \), \( x_k = k\pi/n \), and \( \Delta x = x_k - x_{k-1} \) (which is \( \pi/n \)). But such a sum is a Riemann sum associated to the integral \( \int_0^{\pi} F(x) \, dx \), using the right-end end points to represent each of the \( n \) subintervals into which the interval \([0, \pi] \) has been divided. Since the limit of the Riemann sum defines the value of the integral, our limit is \( \int_0^{\pi} F(x) \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin(x) \, dx \). We
evaluate an antiderivative using Integration By Parts, to get \(-x \cos(x) + \sin(x) + C\); using
the Fundamental Theorem of Calculus the value of the integral is \(\pi\) and so the original
limit is 1.

(v) As in the first limit we substitute \(u = 1/x\) to get \(\lim_{u \to 0^+} (1 - e^{-u})/u\) and then
use L'Hôpital’s Rule to see the limit equals 1.

2. (10 pts.) A perfectly spherical apple of radius 3 centimeters is centered at the origin.

A worm crawls along the \(x\)-axis, eating every bit of the apple whose distance from the
\(x\)-axis is less than 1 centimeter. Find the volume of the remaining uneaten portion of
the apple.

**ANSWER:** We can calculate the volume with the “method of washers”, that is, the
volume is the integral \(\int_{-3}^{3} A(x) \, dx\) of the cross-sectional area of portion that the worm did
not eat of the slice of the apple at a given \(x\) coordinate. Note that \(A(x) = 0\) when \(x\) is
close to \(\pm 3\); in fact the worm eats the entirety of the slice unless \(|x| \leq \sqrt{8}\). Then, for
\(x\) in this interval, the uneaten portion is an annulus (a “washer”) whose inner radius is
always 1cm and whose outer radius is \(\sqrt{9 - x^2}\). Thus the area \(A(x)\) of the uneaten slice is
\(\pi(9 - x^2) - \pi\) cm\(^2\). It follows that the volume of the uneaten portion is

\[
\pi \int_{-\sqrt{8}}^{\sqrt{8}} (8 - x^2) \, dx = \frac{64\sqrt{2}\pi}{3}\text{cm}^3
\]

The volume can also be computed by the method of cylindrical shells.

3. (10 pts.) Compute \(\int_{0}^{\infty} \frac{1}{(1 + x^2)^3} \, dx\).

**ANSWER:** This is an improper integral, so we must compute an antiderivative and study its
endpoint behaviour. Using the substitution \(x = \tan(\theta)\) the integral becomes \(\int \cos^4(\theta) \, d\theta\),
which we evaluate with the customary trigonometric identities:

\[
\cos(\theta)^4 = \frac{1}{4}(1 + \cos(2\theta))^2 = \frac{1}{4} \left(1 + \cos(2\theta) + \frac{1 + \cos(4\theta)}{2}\right) = \frac{1}{32} (12\theta + 8 \sin(2\theta) + \sin(4\theta))
\]

With several applications of the double-angle formulas, this may be written

\[
\frac{1}{8} \left(3t + 4 \cos(t) \sin(t) + 2 \cot^2(t) \sin(t)\right)
\]
Substituting back \( \sin(t) = x/\sqrt{1 + x^2} \) and \( \cos(t) = 1/\sqrt{1 + x^2} \) gives

\[
\int \frac{1}{(1 + x^2)^{\frac{3}{2}}} \, dx = \frac{1}{8} \left( 3 \arctan(x) + \frac{3x}{1 + x^2} + \frac{2x}{(1 + x^2)^{\frac{3}{2}}} \right)
\]

Taking now the integral over any interval \([0, T]\) and letting \( T \to +\infty \) gives the value of the integral as \( 3\pi/16 \).

4. (10 pts.) Line \( L \) is the intersection of the planes \( 2x + 2y + z = 4 \) and \( x - y - z = 1 \).

There are two spheres of radius 3 which pass through the origin and whose centers lie on \( L \). Find the equations of the spheres.

**ANSWER:** It is easier to use a parametric description of this line. The normal vectors of the two planes are \( \langle 2, 2, 1 \rangle \) and \( \langle 1, -1, -1 \rangle \) respectively; the cross product of these two vectors, namely \( \langle 1, -3, 4 \rangle \), is then parallel to both the planes and hence to their intersection, the line \( L \). Pick any point on the line (say, \( (1, 2, -2) \) ) and add multiples of this vector to it to get a parameterization:

\[
L = \{(1 + t, 2 - 3t, -2 + 4t) \mid t \in \mathbb{R} \}
\]

So now we need only to find the values of \( t \) for which a sphere of radius 3 with such a center passes through the origin, that is, the values of \( t \) for which this point is three units away from \( (0, 0, 0) \). Clearly this happens iff \((1 + t)^2 + (2 - 3t)^2 + (-2 + 4t)^2 = 9\). That’s a quadratic equation with roots \( t = 0 \) and \( t = 1 \). So the two good centers on \( L \) are \((1, 2, -2)\) and \((2, -1, 2)\) (which obviously are indeed a distance of 3 from the origin). Then the spheres are give by the equations

\[
(x - 1)^2 + (y - 2)^2 + (z + 2)^2 = 9 \quad \text{and} \quad (x - 2)^2 + (y + 1)^2 + (z - 2)^2 = 9
\]