1. (20 pts.) Compute the following limits

(i) \( \lim_{n \to \infty} \left( 1 - \frac{2}{n} \right)^{3n} \)

(ii) \( \lim_{x \to 0} x^{-1} \int_{3}^{3+x} \cos(\pi y^2) \, dy \)

(iii) \( \lim_{n \to \infty} \sum_{k=0}^{n} \frac{3^k}{k!} \)

(iv) \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k\pi}{n^2} \sin \left( \frac{k\pi}{n} \right) \)

(v) \( \lim_{x \to \infty} x \left( 1 - e^{-\left(\frac{1}{x}\right)} \right) \)

**ANSWER:**

(i) Exponentiation of real numbers \( a^b \) (with \( a > 0 \)) may be written as \( e^{b \ln(a)} \); since the exponential and logarithm functions are continuous we can then compute \( \lim a^b \) as \( e^{\lim(b \ln a)} \). In our case this requires that we compute \( \lim_{n \to \infty} 3n \ln(1 - (2/n)) \). We may substitute \( n = 1/u \); then we need the limit as \( u \to 0^+ \) of \( 3 \ln(1 - 2u)/u \). With one application of L’Hôpital’s Rule this limit is seen to be \(-6\). So the original limit evaluates to \( e^{-6} \).

(ii) Writing this as \( \lim_{x \to 0} \frac{F(x)}{x} \) we see that this again may be computed using L’Hôpital’s Rule (since clearly the integral \( F(x) \) will vanish when \( x = 0 \)). But \( F'(x) = \cos(\pi(3 + x)^2) \) by the Fundamental Theorem of Calculus, so the limit involved in L’Hôpital’s Rule is simply \( \cos(9\pi) = -1 \).

(iii) This is the limit of the partial sums of an infinite series \( \sum_{k=0}^{\infty} 3^k/k! \). But we recognize this as the Taylor series of the exponential function, evaluated at \( x = 3 \). Hence the value of this limit is \( e^3 \).

(iv) This may be written \( \lim_{n \to \infty} \sum_{k=1}^{n} F(x_k) \Delta x \), where \( F(x) = \frac{1}{\pi} x \sin(x) \), \( x_k = k\pi/n \), and \( \Delta x = x_k - x_{k-1} \) (which is \( \pi/n \)). But such a sum is a Riemann sum associated to the integral \( \int_{0}^{\pi} F(x) \, dx \), using the right-end end points to represent each of the \( n \) sub-intervals into which the interval \([0, \pi]\) has been divided. Since the limit of the Riemann
sum defines the value of the integral, our limit is \( \int_0^\pi F(x) \, dx = \frac{1}{\pi} \int_0^\pi x \sin(x) \, dx \). We evaluate an antiderivative using Integration By Parts, to get \(-x \cos(x) + \sin(x) + C\); using the Fundamental Theorem of Calculus the value of the integral is \( \pi \) and so the original limit is 1.

(v) As in the first limit we substitute \( u = 1/x \) to get \( \lim_{u \to 0^+} (1 - e^{-u})/u \) and then use L’Hôpital’s Rule to see the limit equals 1.

2. (10 pts.) A perfectly spherical apple of radius 3 centimeters is centered at the origin. A worm crawls along the \( x \)-axis, eating every bit of the apple whose distance from the \( x \)-axis is less than 1 centimeter. Find the volume of the remaining uneaten portion of the apple.

**ANSWER:** We can calculate the volume with the “method of washers”, that is, the volume is the integral \( \int_{-3}^{3} A(x) \, dx \) of the cross-sectional area of portion that the worm did not eat of the slice of the apple at a given \( x \) coordinate. Note that \( A(x) = 0 \) when \( x \) is close to \( \pm 3 \); in fact the worm eats the entirety of the slice unless \( |x| \leq \sqrt{8} \). Then, for \( x \) in this interval, the uneaten portion is an annulus (a “washer”) whose inner radius is always 1 cm and whose outer radius is \( \sqrt{9 - x^2} \). Thus the area \( A(x) \) of the uneaten slice is \( \pi(9 - x^2) - \pi \) cm\(^2\). It follows that the volume of the uneaten portion is

\[
\pi \int_{-\sqrt{8}}^{\sqrt{8}} (8 - x^2) \, dx = \frac{64\sqrt{2}\pi}{3} \text{cm}^3
\]

The volume can also be computed by the method of cylindrical shells.

3. (10 pts.) Compute \( \int_0^\infty \frac{1}{(1 + x^2)^3} \, dx \).

**ANSWER:** This is an improper integral, so we must compute an antiderivate and study its endpoint behaviour. Using the substitution \( x = \tan(\theta) \) the integral becomes \( \int \cos^4(\theta) \, d\theta \), which we evaluate with the customary trigonometric identities:

\[
\cos(\theta)^4 = \frac{1}{4} (1 + \cos(2\theta))^2 = \frac{1}{4} \left(1 + 2 \cos(2\theta) + \frac{1 + \cos(4\theta)}{2}\right) = \frac{1}{32} (12\theta + 8 \sin(2\theta) + \sin(4\theta))
\]
With several applications of the double-angle formulas, this may be written

\[ \frac{1}{8} \left( 3\theta + 4\cos(\theta) \sin(\theta) + 2\cot^2(\theta) \sin(\theta) \right) . \]

Substituting back \( \sin(\theta) = x/\sqrt{1 + x^2} \) and \( \cos(\theta) = 1/\sqrt{1 + x^2} \) gives

\[ \int \frac{1}{(1 + x^2)^3} \, dx = \frac{1}{8} \left( 3 \arctan(x) + \frac{3x}{1 + x^2} + \frac{2x}{(1 + x^2)^3} \right) \]

Taking now the integral over any interval \([0, T]\) and letting \( T \to +\infty \) gives the value of the integral as \( 3\pi/16 \).

4. (10 pts.) Line \( L \) is the intersection of the planes \( 2x + 2y + z = 4 \) and \( x - y - z = 1 \).

There are two spheres of radius 3 which pass through the origin and whose centers lie on \( L \). Find the equations of the spheres.

**ANSWER:** It is easier to use a parametric description of this line. The normal vectors of the two planes are \( \langle 2, 2, 1 \rangle \) and \( \langle 1, -1, -1 \rangle \) respectively; the cross product of these two vectors, namely \( \langle 1, -3, 4 \rangle \), is then parallel to both the planes and hence to their intersection, the line \( L \). Pick any point on the line (say, \( (1, 2, -2) \) ) and add multiples of this vector to it to get a parameterization:

\[ L = \{ (1 + t, 2 - 3t, -2 + 4t) \mid t \in \mathbb{R} \} \]

So now we need only to find the values of \( t \) for which a sphere of radius 3 with such a center passes through the origin, that is, the values of \( t \) for which this point is three units away from \( (0, 0, 0) \). Clearly this happens iff \( (1 + t)^2 + (2 - 3t)^2 + (-2 + 4t)^2 = 9 \). That’s a quadratic equation with roots \( t = 0 \) and \( t = 1 \). So the two good centers on \( L \) are \( (1, 2, -2) \) and \( (2, -1, 2) \) (which obviously are indeed a distance of 3 from the origin). Then the spheres are given by the equations

\[ (x - 1)^2 + (y - 2)^2 + (z + 2)^2 = 9 \quad \text{and} \quad (x - 2)^2 + (y + 1)^2 + (z - 2)^2 = 9 \]