ALBERT A. BENNETT CALCULUS PRIZE EXAM Dec. 6, 2016

1. Find all lines tangent to the curve $y = x^3 + 14x^2 + 30x + 72$ which pass through the origin.

ANSWER The line joining the point (x, y) to $(0, 0)$ has slope y/x; the line tangent to the curve at (x, y) has slope dy/dx. So the points of tangency we seek are those having $y = x \cdot (dy/dx)$, i.e.

$$x^3 + 14x^2 + 30x + 72 = x \cdot (3x^2 + 28x + 30) \qquad \text{or} \quad 2x^3 + 14x^2 - 72 = 0$$

The roots of the latter cubic are $x = 2$, $x = -3$, and $x = -6$. (You may estimate the roots from a graph of the curve, or you may look for them using the rational root test. And once any one root is found, the others can be found quickly using polynomial division and the quadratic formula.) The lines are $y = 98x$, $y = -27x$, and $y = -30x$ respectively, at the points of tangency $(x, y) = (2, 196)$, $(-3, 81)$, and $(-6, 180)$.

2. Find, with proof, the absolute maximum of the function

$$f(x) = \frac{1}{1 + |x - 2|} + \frac{4}{1 + |x - 5|}$$

ANSWER As with all things absolute-value, it is easiest to consider separately the cases in which the arguments to the absolute values are positive or negative. For different ranges of the variable x, a function of the form $a/(b + |x - c|)$ will equal one of $a(b \pm (x - c))^{-1}$; in either case its first derivative is $\mp a(b \pm (x - c))^{-2}$ and its second derivative is

$$2a(b \pm (x - c))^{-2} = 2a(b + |x - c|)^{-2}$$

In particular, if a and b are positive, we will have a positive second derivative at all points x (except c, where the first and second derivatives do not exist).

It follows that any sum of such functions, such as our $f(x)$, has a positive second derivative wherever it is differentiable, and hence is concave-up at all points except $x = 2$ and $x = 5$. So these are the only candidates for a maximum of f, and we observe $f(2) = 1/1 + 4/4 = 2$ and $f(5) = 1/4 + 4/1 = 17/4$, so the absolute maximum is attained at $x = 5$.

3. Does the improper integral $\displaystyle\int_0^1 \frac{e^{-1/x}}{x^3}\,dx$ converge?

ANSWER We must decide whether the proper integrals $\displaystyle\int_t^1 \frac{e^{-1/x}}{x^3}\,dx$ have a limit as $t \to 0$ from above. To compute this integral, let $u = 1/x$ to transform it to $\int_1^T ue^{-u}\,du$ where $T = 1/t$. This integral we evaluate using integration by parts and the Fundamental Theorem: an antiderivative is $-(u+1)e^{-u}$ and so the integral equals $2/e - (T+1)/e^T$. As t decreases to zero, T increases to $+\infty$ and the integral converges to $2/e$ by (for example) L'Hopital's Rule. So yes, the improper integral converges (to $2/e$).

4. What is the sum of this series?

$$\sum_{n=1}^{\infty} \left(\frac{(-3)^n - n}{4^n} \right)$$

ANSWER The sum may be broken into two parts: it's $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{4^n} - \sum_{n=1}^{\infty} \frac{n}{4^n}$. The first part is a geometric series with initial term $a = -3/4$ and common ratio $r = -3/4$ as well; it is then well known that the sum is $a/(1-r) = (-3/4)/(7/4) = -3/7$.

To evaluate the second sum, recall that for every x of magnitude less than 1, we have $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$. By either squaring both sides or differentiating both sides with respect to x, or by computing the Maclaurin series of the right-hand side, we deduce that

$$\sum_{n=0}^{\infty} n\,x^{n-1} = (1-x)^{-2}$$

The zero$^{\text{th}}$ term on the left is zero and may be ignored. If we now set $x = 1/4$ in this infinite series and multiply by one additional factor of $1/4$ we obtain the other sum in the original problem, which therefore evaluates to $(1/4)(1-1/4)^{-2} = 4/9$.

Thus the original series sums to $-3/7 - 4/9 = -55/63$.

5. For this function of two variables

$$f(x,y) = \begin{cases} xy(x^2-y^2)/(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases},$$

compute $f_{yx}(0,0) - f_{xy}(0,0)$ (that is, the value of $\dfrac{\partial^2 f}{\partial x \partial y} - \dfrac{\partial^2 f}{\partial y \partial x}$ at the origin.)

ANSWER It is straightforward to compute the partial derivatives away from the origin:

$$f_x(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} \qquad f_y(x,y) = \frac{x(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}$$

We may also compute the partial derivatives at $(0,0)$ itself using the definition of a derivative, e.g. $f_x(0,0) = \lim_{h\to 0}(f(h,0) - f(0,0))/h$. But both terms in the numerators are zero, so the partial derivatives are as well.

Then we may compute the second derivatives. As is well known, the partials $f_{xy}(p)$ and $f_{yx}(p)$ will be equal at all points p other than the origin; there is no need to do the algebra to compute $f_{xy}(p)$. But at the origin we must again resort to the definition, computing

$$\begin{aligned} f_{yx}(0,0) &= \lim_{h\to 0}(f_y(h,0) - f_y(0,0))/h = \lim_{h\to 0}((h^5/h^4) - 0)/h = +1 \\ f_{xy}(0,0) &= \lim_{h\to 0}(f_x(0,h) - f_x(0,0))/h = \lim_{h\to 0}((-h^5/h^4) - 0)/h = -1 \end{aligned}$$

so the requested difference is $+2$.

The fact that, in general, $f_{xy}(p) = f_{yx}(p)$ is a theorem, and so is only true when some conditions are met. The usual condition used is that the second-order mixed partial derivatives must be continuous. (This is often called Clairaut's Theorem or Young's Theorem.) In this particular example (which is considered prototypical), along any line through the origin we find $f(x,y)$ to be proportional to x^2, hence the first derivatives are proportional to x and the second derivatives are constant. Indeed in polar coordinates

$$f_{xy}(r\cos(\theta), r\sin(\theta)) = -16\cos(\theta)^6 + 24\cos(\theta)^4 - 6\cos(\theta)^2 - 1$$

This function is continuous everywhere except at the origin ($r = 0$), where f_{xy} is discontinuous.