1. The equation \( x^y = y^x \) describes a curve in the first quadrant of the plane containing the point \( P = (4, 2) \). Compute the slope of the line that is tangent to this curve at \( P \).

**ANSWER:** Take logarithms to see that this equation is equivalent to \( \ln(x)/x = \ln(y)/y \), that is, \( f(x) = f(y) \) where \( f(x) = \ln(x)/x \).

Now use Implicit Differentiation to get \( f'(x) = f'(y) \frac{dy}{dx} \), i.e.

\[
\frac{dy}{dx} = \frac{f'(x)}{f'(y)} = \frac{(1 - \ln(x))/x^2}{(1 - \ln(y))/y^2}.
\]

At the point \( P \) this gives a value of

\[
\frac{dy}{dx} = \frac{(1 - 2 \ln(2))/16}{(1 - \ln(2))/4},
\]

about \(-0.31\).

If you know it, you may also use the Implicit Function Theorem, which tells us directly that on the curve \( F(x, y) = 0 \) we have

\[
\frac{dy}{dx} = -\left( \frac{\partial F}{\partial y} \right)^{-1} \left( \frac{\partial F}{\partial x} \right)
\]

You could also use Implicit Differentiation directly on the equation \( x^y = y^x \); it is helpful to recall that when \( a \) is independent of \( x \), we have \( (d/dx)(a^x) = a^x \cdot \log(a) \).

The graph of this function \( f \) is very revealing: \( f(x) \) is positive iff \( x > 1 \), increases to a maximum of \( 1/e \) at \( x = e \), and then decreases to zero as \( x \to \infty \). Thus \( f \) achieves each value in \((0, 1/e)\) precisely twice. Hence \( f(x) = f(y) \) iff \( x = y \) or else \( x \) and \( y \) are such a pair of numbers where \( f \) attains a single value — one of them in \((1, e)\) and the other in \((e, \infty)\). Note that as \( x \to 1^+, y \to \infty \), and vice versa.

So the graph of \( x^y = y^x \) consists of the line \( y = x \) and this set of pairs \((x, y)\) with \( f(x) = f(y) \). It has asymptotes at \( x = 1 \) and at \( y = 1 \) and is contained in the region where \( x > 1 \) and \( y > 1 \). The formula above for \( dy/dx \) then shows \( dy/dx \) is everywhere negative, so the graph is everywhere decreasing.
2. Determine whether this series is convergent or divergent:

\[ \sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}} \]

**ANSWER:** The series diverges.

Each term is positive, and may be written as \( e^{-(\ln(ln(n)))^2} \). Now, for all \( x > 0 \) we have \( \ln(x) < \sqrt{x} \). (This is easily proved by noting that \( f(x) = \sqrt{x} - \ln(x) \) has a minimum value at \( x = 4 \), where \( f(4) = 2\ln(e/2) > 0 \).) So \( (\ln(x))^2 < x \) for all \( x > 0 \), including when \( x = \ln(n) \), i.e. \( e^{(\ln(ln(n)))^2} < e^{\ln(n)} = n \). (Equivalently, let \( y = \ln(ln(n)) \) and then note \( (\ln(n))^{\ln(ln(n))} = (e^y)^y = e^{(y^2)} < e^{(e^y)} = n \).) Thus the \( n \)th term of our series is greater than \( 1/n \), and hence this series diverges by comparison to the Harmonic Series.

(For comparison, \( \sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}} \) converges, this one being the sum of \( 1/n^{\ln(ln(n))} \), and that exponent is more than 1.019 > 1 for all \( n > 15 \).)

3. Compute \( \int_{0}^{\pi/4} \frac{1}{\cos(x) + \sin(x)} \, dx \).

**ANSWER:** The integral will be computed to be \( \ln(\sqrt{2} + 1)/\sqrt{2} \) by the Fundamental Theorem of Calculus. We may use any of several substitutions to help us find the antiderivative.

Simplest, perhaps, is to note that

\[ \cos(x) + \sin(x) = \sqrt{2} \left( \cos \left( \frac{\pi}{4} \right) \cos(x) + \sin \left( \frac{\pi}{4} \right) \sin(x) \right) = \sqrt{2} \cos \left( x - \frac{\pi}{4} \right) \]

so if we use the substitution \( u = x - (\pi/4) \) our integral becomes

\[
\int_{u=-\pi/4}^{0} \frac{du}{\sqrt{2} \cos(u)} = \frac{1}{\sqrt{2}} \int_{u=-\pi/4}^{0} \sec(u) \, du = \frac{1}{\sqrt{2}} \ln (|\sec(u) + \tan(u)|) \bigg|_{-\pi/4}^{0} = -\frac{1}{\sqrt{2}} \ln(\sqrt{2} - 1) = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)
\]

(More generally any linear combination of \( \sin(x) \) and \( \cos(x) \) may be combined into a single term \( A \cos(x + B) \) in a similar way.)

A closely-related suggestion would be \( u = \sin(x) - \cos(x) \); then \( u^2 = 1 - 2 \sin(x) \cos(x) \) and \( dx/(\cos(x) + \sin(x)) = du/(2 - u^2) \), which we must integrate from \( u = -1 \) to \( u = 0 \), to get the same conclusion.
Or we may use the half-angle substitution \( u = \tan(x/2) \). I will let you check the trigonometry that proves

\[
\cos(x) = \frac{1 - u^2}{1 + u^2}, \quad \sin(x) = \frac{2u}{1 + u^2}, \quad dx = \frac{2 \, du}{1 + u^2}
\]

This transforms our integral into

\[
\int_{u=0}^{u=\sqrt{2}-1} \frac{1 + u^2}{(1 - u^2) + (2u)} \frac{2 \, du}{1 + u^2} = -2 \int_{u=0}^{u=\sqrt{2}-1} \frac{du}{(u - 1)^2 - 2}
\]

which may be computed using Partial Fractions. (This half-angle substitution converts any rational function of the six trig functions into a rational function of \( u \), which may then be computed using Partial Fractions.)

Or we may use de Moivre’s formulas: \( \cos(x) = (e^{ix} + e^{-ix})/2 \) and \( \sin(x) = (e^{ix} - e^{-ix})/(2i) \). In this case we may let \( u = e^{ix} \) (so \( du = iu \, dx \)) and the substitution results in a rational function of \( u \), again amenable to Partial Fractions.

Or, multiply numerator and denominator by \( \cos(x) - \sin(x) \) to write the integrand as

\[
\frac{\cos(x)}{1 - 2 \sin^2(x)} + \frac{-\sin(x)}{2 \cos^2(x) - 1}
\]

(Each denominator equals \( \cos^2(x) - \sin^2(x) \).) Substitute \( u = \sin(x) \) resp. \( u = \cos(x) \), to transform each of the two indefinite integrals to \( \pm \int \frac{du}{1 - 2u^2} \), which can be evaluated using Partial Fractions and then rewritten again in terms of sines and cosines. (Some care must be taken when evaluating this antiderivative at \( x = \pi/4 \) precisely because the extra factor \( \cos(x) - \sin(x) \) which we introduced vanishes at this value of \( x \), that is, we have created an improper integral. In that case we take a limit as the upper limit of integration approaches \( \pi/4 \) from the left, using L’Hôpital’s Rule in the antiderivative.)

This last approach illustrates another general technique useful for trig integrals. What we did here was to write the integrand \( f(x) \) as the sum of its even part \((f(x) + f(-x))/2 \) and its odd part \((f(x) - f(-x))/2 \); the first is a function \( f_1(\cos(x)) \) of \( \cos(x) \) alone and the second may similarly be written as \( \sin(x) f_2(\cos(x)) \). For the second integral we use the substitution \( u = \cos(x) \) to get \( \int f_2(u) \, du \), that is we have removed the trigonometry. For the first integral, we first write \( f_1 \) in terms of its even and odd parts, one being a function \( f_3(\cos^2(x)) \) of \( \cos^2(x) \) alone and the other being of the form \( \cos(x) f_4(\cos^2(x)) = \cos(x) f_4(1 - \sin^2(x)) \); of these two parts the first is improved with the double-angle formula for cosine, and the second is amenable to the substitution \( u = \sin(x) \).
4. A wedding ring is the three-dimensional solid that remains after drilling a cylindrical hole through the center of a sphere. Compute, with proof, the volume of metal in a metallic wedding ring that is 6mm tall when it rests on a table, as a function of the radius $r$ of the hole that has been drilled.

**Answer:** When $r = 0$, we have drilled out none of the sphere, so the height is the diameter of the sphere, and the volume is the volume of a sphere of radius 3mm, namely $V = \frac{4}{3}\pi r^3 = 36\pi$ mm$^3$. Amazingly, the volume of metal is $36\pi$ mm$^3$ irrespective of the value of $r$!

Let $s$ be the radius of the sphere, i.e. the distance from a point on the outside of the wedding ring to the center of the sphere. If that point is at a distance $d$ from the central axis of the wedding ring, and at a height $h$ above the center of the sphere, then a small right triangle will show that $s^2 = h^2 + d^2$. In the special case that the point is along the top edge of the wedding ring, this shows $s^2 = 3^2 + r^2$.

Now, this “wedding ring” can be viewed as a solid of revolution; we are spinning a region shaped like a thin letter $D$ around the central axis. Thus the cross-section perpendicular to the central axis at a height of $h$ is an annulus (“washer”) with an inner radius of $r$ and an outer radius of $d$, and hence of area $\pi d^2 - \pi r^2 = \pi(s^2 - h^2) - \pi(s^2 - 3^2) = \pi(9 - h^2)$.

Thus the volume of the ring is, as advertised,

$$\int_{h=-3}^{h=+3} \pi(9 - h^2) \, dh = \pi \left( 9h - \frac{h^3}{3} \right) \bigg|_{-3}^{3} = 36\pi.$$

One may also use the method of shells: at a distance $t$ from the central axis ($0 < t < s$) the half of the figure lying above the central plane is a cylinder of height $\sqrt{s^2 - t^2}$ (and radius $t$), hence the total volume of the top half of the ring is $\int_r^s 2\pi t \sqrt{s^2 - t^2} \, dt = (2\pi/3)(s^2 - r^2)^{3/2}$, and yet as noted above $s^2 - r^2 = 9$mm$^2$, giving the same conclusion as above. One obtains precisely the same integral by computing the volume as an integral $2 \int_A \sqrt{s^2 - (x^2 + y^2)} \, dx \, dy$ over the annulus $A$ having radii $r$ and $s$, when using polar coordinates.

This is a well-known question.
5. The curve parameterized by $x(t) = \cos^3(t)$, $y(t) = \sin^3(t)$, $z(t) = \cos(2t)$ passes through the point $(1, 0, 1)$ when $t = 0$ and passes through the point $(0, 1, -1)$ when $t = \pi/2$, having traversed a path of length $5/2$. (You don’t have to prove this.) What point will it pass through after having traversed a length of exactly 1?

**ANSWER:** The length of a segment of a parameterized curve can be viewed as the distance traveled, and distance is the integral of speed (over time); speed is the magnitude of the velocity vector, and velocity is the rate of change of position. That is, the length of the curve between $t = a$ and $t = b$ is

$$L = \int_{t=a}^{t=b} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt$$

In our case $x'(t) = -3\cos^2(t)\sin(t)$ and $y'(t) = 3\sin^2(t)\cos(t)$, so already we have $(x')^2 + (y')^2 = (3\sin(t)\cos(t))^2 (\cos^2 t + \sin^2 t) = 9\sin^2(t)\cos^2(t)$. Next, since $z'(t) = -2\sin(2t) = -4\sin(t)\cos(t)$, we then have $(x')^2 + (y')^2 + (z')^2 = (\sin(t)\cos(t))^2 (3^2 + 4^2)$, so the integrand is the square root of this: $5|\sin(t)\cos(t)|$. In our case we will have $t \in (0, \pi/2)$, so both the sine and the cosine are positive and the absolute value bars unnecessary. Thus

$$L = \int_{t=0}^{t=b} 5\sin(t)\cos(t) \, dt = \frac{5}{2} \sin^2(t) \bigg|_0^b = \frac{5}{2} \sin^2(b)$$

This is equal to 1 when $\sin(b) = \sqrt{2/5}$. Then $\cos(b) = \sqrt{3/5}$ and $\cos(2b) = \cos^2(b) - \sin^2(b) = 1/5$, so the point we will have reached is

$$(x, y, z) = \left((3/5)^{3/2}, (2/5)^{3/2}, 1/5\right).$$

Of course there is nothing special about making the length equal to 1; we will have traversed any distance $\ell < 5/2$ when $\sin^2(b) = 2\ell/5$ so that we will be at position

$$p(\ell) = \left((1 - \frac{2\ell}{5})^{3/2}, \left(\frac{2\ell}{5}\right)^{3/2}, 1 - \frac{4\ell}{5}\right).$$

This function $p$ provides a new parameterization of the same curve, with now the additional feature that for all $t$, the curve from $p(0)$ to $p(t)$ has arclength $t$, that is, we are traversing the curve at a constant speed of 1. This kind of re-parameterization is always possible (at least in principle, if not with formulas) for any smooth curve; it is called **parameterization with respect to arclength**.