ALBERT A. BENNETT CALCULUS PRIZE EXAM May 4 2013

1. For each real number a, evaluate $\int_0^\infty e^{ax}\cos(x)\,dx$ or explain why the integral diverges.

ANSWER The antiderivative $I = \int e^{ax}\cos(x)\,dx$ may be determined by two applications of integration-by-parts: we discover

$$I = e^{ax}\sin(x) - a\int e^{ax}\sin(x)\,dx$$

but similarly we find

$$\int e^{ax}\sin(x)\,dx = -e^{ax}\cos(x) + a\int e^{ax}\cos(x)\,dx$$

so that

$$I = e^{ax}\sin(x) + a\,e^{ax}\cos(x) - a^2 I$$

from which we deduce (as can be checked by differentiating!) that

$$(a^2+1)I = e^{ax}(a\cos(x) + \sin(x))$$

Thus the integral from 0 to T is

$$\frac{1}{(a^2+1)}\left(e^{aT}(a\cos(T) + \sin(T)) - a\right)$$

For negative a, the exponential factor decreases to zero as T increases, while the other factors stay bounded, so the improper integral converges to $\frac{-a}{(a^2+1)}$. When $a = 0$ the integral oscillates with T so no limit exists, and likewise when $a > 0$ the magnitudes are unbounded and so the improper integral diverges.

2. Evaluate $\sum_{n=2}^\infty \frac{1}{(n-1)(n+2)}$ or explain why the series diverges.

ANSWER Using partial fractions (say) we discover that $\frac{1}{(n-1)(n+2)}$ may be written $\frac{1/3}{(n-1)} - \frac{1/3}{(n+2)}$. Writing out the first few terms we recognize a telescoping sum:

$$\left(\frac{1/3}{1} - \frac{1/3}{4}\right) + \left(\frac{1/3}{2} - \frac{1/3}{5}\right) + \left(\frac{1/3}{3} - \frac{1/3}{6}\right) + \left(\frac{1/3}{4} - \frac{1/3}{7}\right) + \dots$$

When we cancel corresponding parts only $(1/3)\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right)$ remains, so the sum is $11/18$.

3. The polar equation of the curve shown in the attached figure is $r = e^{-\theta/10}$. Assume that the pattern of shaded and unshaded sections continues ad infinitum. What is the area of the shaded region? Simplify your answer as much as possible.

ANSWER The figure shows a sequence of arcs that look nearly like semicircles in the top half of the plane. We can get the area of the shaded region as an alternating sum: it's the area beneath the outermost curve, minus the area under the second curve, plus the area under the third, and so on. It is clear from the geometry that these terms are steadily decreasing in magnitude and tend to zero, so this alternating sum will converge. To get the actual sum, we need only get an expression for the area of the region beneath each of these curves.

We can find the area inside any of those curves with an integral $\frac{1}{2}\int r^2\,d\theta$. The integrand is $e^{-\theta/5}$, whose anti-derivative is $-5e^{-\theta/5}$. Thus the area inside the outermost curve is $\frac{1}{2}(-5e^{-\pi/5} + 5e^{-0/5})$ and similarly the area inside the next arc is $\frac{1}{2}(-5e^{-3\pi/5} + 5e^{-2\pi/5})$ and so on. Writing X for $e^{-\pi/5}$, the successive areas are thus $\frac{5}{2}(1-X)$, $\frac{5}{2}(X^2 - X^3)$, $\frac{5}{2}(X^4 - X^5), \ldots$ Observe that each of these terms is X^2 as large as the one before it.

Therefore, we find the total area of all the shaded regions to be the alternating sum

$$\frac{5}{2}(1-X)(1-X^2+X^4-X^6+\ldots) = \frac{5}{2}(1-X)/(1+X^2),$$

using the formula for the sum of a geometric series. That is, the area is

$$5(1-e^{-\pi/5})/2(1+e^{-2\pi/5}).$$

(The numerical value of this is approximately 0.908 — a little more than half the area $\pi/2$ of the upper unit half-disk, which seems visually appropriate.)

4. Here are four lines in space:

$$L_1 : \{x = 1, y = 0\} \qquad L_2 : \{y = 1, z = 0\} \qquad L_3 : \{z = 1, x = 0\} \qquad L_4 : \{x = y = -6z\}$$

For partial credit, find a line that intersects both L_1 and L_2. For full credit, find a line that intersects all three of L_1, L_2, and L_3. For extra credit, find a line that meets all four of the lines L_i.

ANSWER For the first task you need only pick any points $P_1 \in L_1$ and $P_2 \in L_2$ and construct the line that joins them. For example if $P_1 = (1, 0, 0)$ and $P_2 = (0, 1, 0)$ then we construct the line $\{x + y = 1, z = 0\}$ in the x, y plane. (Some students also noted that we can simply combine the non-contradictory conditions used to *define* L_1 and L_2 in the first place: the line $\{x = 1, z = 0\}$ meets both L_1 and L_2.)

For the second task we don't have so much liberty — for example the line we have just constructed in the plane $z = 0$ does not intersect the line L_3 in the parallel plane $z = 1$. But we might yet succeed if we choose different points of intersections with the first two lines. Writing $P_1 = (1, 0, a)$ and $P_2 = (b, 1, 0)$ for those two points, the line we construct may be written in parametric form as the set of points

$$P_1 + t(P_2 - P_1) = (1, 0, a) + t(b - 1, 1, -a)$$

In order for this line to meet L_3 there must be a value of t that makes $z = a - ta$ equal to 1 and $x = 1 + t(b - 1)$ equal to zero. The first condition requires $at = a - 1$ and the second requires $(b - 1)t = -1$, and these conditions are contradictory unless $(b - 1)(a - 1) = -a$ (and $b \neq 1$ and $a \neq 0$). In short, we can still pick P_1 to be any point on the first line *except* $(1, 0, 0)$ or $(1, 0, 1)$, and then compute a matching P_2 and build the line joining P_1 and P_2 as before. If for example we use $a = -1$ we need $b = 1/2$, giving us the line expressed parametrically as

$$(1, 0, -1) + t(-1/2, 1, 1)$$

which crosses L_3 when $t = 2$, i.e. at the point $P_3 = (0, 2, 1)$. (You can also approach the problem a bit more symmetrically: you need to find three collinear points $(1, 0, a), (b, 1, 0)$, and $(0, c, 1)$; it turns out they are collinear iff $b(a - 1) = c(b - 1) = a(c - 1) = abc = -1$. There are infinitely many triples (a, b, c) that satisfy these equations.)

Finally in order to meet the line L_4 too, we obtain *another* constraint on a and b in the same way: the line joining P_1 and P_2 will meet L_3 only if $(b-1)(a-1) = -a$ as before, and likewise we find it will meet L_4 only if there is a time t' when the point

$$(x, y, z) = (1, 0, a) + t'(b-1, 1, -a) = (1+t'(b-1), t', a(1-t')) = (1-at'/(a-1), t', a(1-t'))$$

satisfies $x = y = -6z$. The first equation holds iff $t' = (a-1)/(2a-1)$ and the second holds iff $t' = 6a/(6a-1)$; these two equations are consistent iff $(a-1)(6a-1) = 6a(2a-1)$, i.e. $6a^2 + a - 1 = 0$, a quadratic equation whose roots are $a = 1/3$ and $a = -1/2$. In each case we compute b using the constraint imposed by line L_3: $b = 1/2$ resp. $b = 2/3$. Then we can draw our lines: $(x, y, z) = (1 + t/2, t, (1-t)/3)$ or $((2+t)/3, 1-t, -t/2)$ It is easily confirmed that each of these two lines meets each of the four L_i.

In summary, there are exactly TWO lines meeting each of the four given lines L_i. This is true in general: given a "generic" set of four lines in 3-space there are exactly two lines that meet all four of them. (However, this statement only becomes precise when we allow for complex numbers and "points at infinity".) You might enjoy extending the computations used above to cover the general case!

5. Find a differentiable function $f(x, y)$ defined in the first quadrant of the plane which has this property: at each point (x, y) the gradient $\nabla f(x, y)$ is perpendicular to the vector $\langle x, y \rangle$ pointing directly away from the origin.

For extra credit: Is there such a function f defined on all of the plane?

ANSWER Of course a constant function f meets the desired condition! That's not what was intended, however, so let's think about non-constant f.

The given condition is equivalent to requiring the directional derivative $D_v(f)$ to vanish where v is the vector pointing away from the origin. It follows that f will stay constant on rays from the origin, so that f depends only on the polar coordinate θ. So for example we may take $f(x, y) = \tan(\theta) = y/x$; it is easily checked that the gradient vector in this case, which is $\langle -y/x^2, 1/x \rangle$ is perpendicular to $\langle x, y \rangle$.

But θ is only continuous on quadrants, or at most on a portion of the plane which excludes some curve stretching from the origin to infinity. Indeed, since the gradient points in the direction of greatest increase, it is never possible for a closed curve to follow the gradient of any function – how can you keep climbing uphill and yet return to your start? In our case, any circle centered at the origin would be such a curve, so no such f can exist.