2016 Texas Putnam Prep Group — week 1, selected answers

My hope is to have students write up answers each week for whatever problems they solved. That way we can learn from each other and critique each other's presentations. This first week there was a solver for each problem except the last, so let me just make a few comments and then show a solution to the last problem.

1. Starting with the input of 11, the sequence of numbers $\{f_n(11)\}$ eventually reached the cycle $\{4, 16, 37, 58, 89, 145, 42, 20, 4, \ldots\}$. What happens with other inputs? The first thing to observe is that for almost every natural number k we have $f(k) < k$. Indeed, if k is a d-digit number then $f(k) < 81d$ whereas k itself is larger than the smallest d-digit number, which is 10^{d-1}. I leave it to you to check that the function $f(x) = 10^x - 810x$ is increasing on the interval $[3, \infty)$ and it is positive at $x = 4$, so for all $d \geq 4$ we have $10^{d-1} > 81d$ and thus $k > f(k)$ for all numbers k of 4 or more digits. A more careful analysis will show that $f(k) < k$ for 3-digit numbers as well, or of course you could simply check them each by hand. This means that for any $k \geq 100$ the sequence of numbers $\{f_n(k)\}$ will begin with a decreasing sequence of numbers until eventually it includes a number less than 100. To determine the fate of such a sequence we then need only investigate what happens to these sequences in each of the cases $k = 1, 2, \ldots 99$. I used a computer to slog through these cases for me: it shows that in every single case by the time we compute $f_{10}(k)$ we have either obtained one of the 8 cycling values or we have reached 1 (and then of course $f_n(k) = 1$ for all further values of n). The slowest sequences for $k < 100$ were for $k = 6$ and $k = 60$. (The slowest convergence to 1 for $k < 100$ occurs for $k = 7$: we get $7, 49, 97, 130, 10, 1, \ldots$)

2. It's actually quite obvious that for every n and for every $k \leq n$ there is a unique decomposition of n of the type requested having exactly k summands. Just imagine trying to put n pennies into k piles with the requirement that the piles all be of nearly the same size. You would could out the pennies, one per pile, until each pile has one penny; then put a second penny into each pile, etc. At the end you put the last few pennies each onto one of the leading piles until you run out of pennies.

Mathematically you simply quote the Division Algorithm: there is a unique pair of integers q and r with $n = kq + r$ and with $0 \leq r < q$. Then the k piles will all have q pennies leaving r left over to go, one each, onto r taller piles. This gives a decomposition of n as the sum of k qs and $(q + 1)$s, there being r of the latter.

Since we have n choices for what k can be, $c(n) = k$.

As with evey other problem involving the phrase "for every natural number n", you should try to prove the result for some small values of n before jumping into the general case. See if you believe there are 13 different decompositions of the number 13, one for each possible number of summands.

6. We must understand the word "cubic" to mean the polynomial has a nonzero coefficient of x^3, as parabolas and lines never cross any line three times. I can think of several things to do.

You might try to reduce your cubic to a normal form. It's easy to check that every cubic has a unique inflection point; translate that point to the origin and scale your axes suitably and you will find the cubic is equivalent to either $y = x^3 + x$, $y = x^3$, or $y = x^3 - x$. Each cubic $y = x^3 + cx$ intersects the line $y = (c+1)x$ where $x = 0$, $x = -1$, or $x = 1$.

You can also just draw a line through any point: as long as the line is more sloped than the tangent line at your chosen point, then it will necessarily cross the cubic again, once to the right of your chosen point and once to the left, since cubics tend to infinity faster than linear functions.

For example, if the cubic is $y = ax^3 + bx^2 + cx + d$, you might pass a line through the point $(0, d)$ with slope $c + a$. That line $y = (c+a)x + d$ meets the cubic at $x = 0$ and where x is either root of $ax^2 + bx - a$.

7. Find an odd prime factor of a_{2015}, where $a_0 = 1$, $a_1 = 2$, and for $n > 1$, $a_n = 4a_{n-1} - a_{n-2}$.

Answer: I claim that a_m divides a_n whenever m divides n and n/m is odd. Therefore a_{2015} will be divisible by (among others) $a_5 = 362 = 2 \cdot 181$, where 181 is easily checked to be prime. So we will be done as soon as I prove this surprising claim.

This is a sequence determined by a linear recurrence relation and hence it is possible for us to obtain explicit formulas for the terms in the sequence. The usual device is the use of "generating functions". Let $G(T)$ be the formal power series

$$G(T) = \sum_{n \geq 0} a_n T^N = a_0 + a_1 T + a_2 T^2 + \dots$$

This series converges in a neighborhood of $T = 0$ but that is not relevant for us. It is a consequence of the linear recurrence relation that almost all the terms of the power series $(1 - 4T + T^2) \cdot G(T)$ vanish, and in fact we can multiply out the first few terms to see that

$$(1 - 4T + T^2) \cdot G(T) = 1 - 2T$$

Therefore the power series $G(T)$ must be that of the rational function $(1-2T)/(1-4T+T^2)$, which we can evaluate using partial fractions:

$$G(T) = (1 - 2T)/(1 - 4T + T^2) = \frac{A}{1 - rT} + \frac{B}{1 - sT}$$

where $r, s = 2 \pm \sqrt{3}$ are the roots of the quadratic and where we compute $A = B = 1/2$. This makes it easy to expand $G(T)$ as a sum of two geometric series, and so we conclude

$$a_n = \frac{1}{2}(r^n + s^n) \ .$$

Now, if say $n = mk$, then let $R = r^m$, $S = s^m$. Then we have

$$a_m = \frac{1}{2}(R + S) \qquad a_n = \frac{1}{2}(R^k + S^k)$$

and so if k is odd,

$$a_n/a_m = R^{k-1} - SR^{k-2} + S^2 R^{k-3} - \dots - RS^{k-2} + S^{k-1}$$

This right-hand side is an algebraic integer and the left-hand side is rational, so the number in question is both of these and hence is an ordinary integer, i.e. a_m divides a_n, as claimed.

Remark 1: The number a_{2015} is the product of the primes

$$2, 181, 373, 193441, 6811741, 150719356321, 1215497709121, 360250962984637,\\ 28572709494917432101, 3045046274679316654761356161,\\ 13277360555506179816997827126375881581$$

and a composite cofactor of 1014 digits that I did not succeed in factoring completely, but it has the 201-digit composite divisor

$$44124624968377630753466527707036141486432876000792101115312050381009867981449539478070999160403047906\\ 2437901343723158398371277010947002579536100391437521801011117135414630111807400169179873321161227361$$

if you'd like to give that a whirl!

Remark 2: All these comments about the given sequence are familiar to people who have asked corresponding questions about the Fibonacci sequence.