UT Putnam Prep Problems, Nov 16, 2016

You asked for an assortment of problems of different types. All righty, then; here's a grab-bag of unrelated questions.

1. Find all solutions in real numbers x, y, z, w to the system

$$x + y + z = w, \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}$$

2. Let M be a finite set of points in the plane such that for any two of them, there is a third point in M on the same line. Show that M is contained in a single straight line.

3. Let $f(x) = \sum_{i=0}^{i=n} a_i x^{n-i}$ be a polynomial of degree n with integral coefficients. If a_0, a_n, and $f(1)$ are odd, prove that $f(x) = 0$ has no rational roots.

4. Define a sequence of real numbers r_n by the recurrence

$$\begin{aligned} r_1 &= 1 \\ r_{n+1} &= 1 + \tfrac{n}{r_n} \end{aligned}$$

Show that for every n, $\sqrt{n} \le r_n \le 1 + \sqrt{n}$.

5. Let g be a continuous function mapping a real interval J into itself. Suppose that there is an integer $n \ge 2$ for which the n-fold composite $g \circ g \circ g \ldots \circ g = I$, the identity functions. (That is, $I(x) = x$ for all $x \in J$.) Prove that $g \circ g = I$.

6. In a round-robin tournament with players $P_1, \ldots, P_n$, each player plays every other player exactly once, and there are no ties. Let w_r and ℓ_r denotes the numbers of games won and lost, respectively, by player r. Prove that

$$\sum_{r=1}^{n} w_r^2 = \sum_{r=1}^{n} \ell_r^2$$

7. The polynomials $P(z)$ and $Q(z)$ have complex coefficients. $P(z)$ and $Q(z)$ have precisely the same sets of zeros, possibly with different multiplicities. $P(z)+1$ and $Q(z)+1$ likewise have the same zeros, possibly with different multiplicities. Prove that $P = Q$.

8. Suppose the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

has three different solutions $y_1(x), y_2(x), y_3(x)$, each defined on the whole real line, such that

$$y_1(x)^2 + y_2(x)^2 + y_3(x)^2 = 1$$

for all real x. Let

$$f(x) = y_1'(x)^2 + y_2'(x)^2 + y_3'(x)^2$$

Find constants A and B so that f is a solution of

$$y' + Ap(x)y = Br(x)$$