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Putnam Solutions

Problem 1

Proof. We compute the first few terms of the sequence as follows:

$$\begin{aligned}
f_0(11) &= 11 \\
f_1(11) &= 1^2 + 1^2 = 2 \\
f_2(11) &= 2^2 = 4 \\
f_3(11) &= 4^2 = 16 \\
f_4(11) &= 1^2 + 6^2 = 37 \\
f_5(11) &= 3^2 + 7^2 = 58 \\
f_6(11) &= 5^2 + 8^2 = 89 \\
f_7(11) &= 8^2 + 9^2 = 145 \\
f_8(11) &= 1^2 + 4^2 + 5^2 = 42 \\
f_9(11) &= 4^2 + 2^2 = 20 \\
f_{10}(11) &= 4 \\
&\vdots
\end{aligned}$$

and we see the sequence is eventually periodic. In particular if $n \geq 10$ we have $f_n(11) = f_{n-8}(11)$. Thus $f_{2016}(11) = f_8(11) = 42$. □

Problem 2

Proof. Note that the problem is equivalent to finding the number of triples (k, a, r) such that $n = ka_1 + r$ where $0 \leq r \leq k - 1$. To see this take a sequence $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$ and note that if $n = a_1 + a_2 + \cdots + a_k$ then $n = ka_1 + r$ where $0 \leq r \leq k - 1$ with $r = (a_2 - a_1) + \cdots + (a_k - a_1)$. On the other hand if we have a triple (k, a, r) such that $n = ka_1 + r$ with $0 \leq r \leq k - 1$ define $a_i = a$ for $1 \leq i \leq k - r + 1$ and $a_i = a + 1$ for $k - r + 1 \leq i \leq k$. Then $a_1 + a_2 + \cdots a_k = (k - r + 1)a + (k - k + r)(a + 1) = ka + r = n$. Thus we have a 1-to-1 correspondence between the desired decompositions $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$ and the defined triples (k, a, r). However the number of such triples is n: for each $1 \leq k \leq n$ take

$r = n \mod k$ and $a = \frac{n-r}{k}$ and note $n = ak + r$ and $0 \leq r \leq k-1$. Thus we have $c(n) = n$. $\square$

Problem 3

Proof. Note $1 + x \leq 2^0(1 + x)$ trivially and $2(1 + xy) - (1 + x)(1 + y) = 2 + 2xy - 1 - x - y - xy = 1 - x - y + xy = (1 - x)(1 - y) > 0$ since $x, y \in [0, 1]$ which implies $(1+x)(1+y) \leq 2(1+xy)$. We proceed to prove the statement by induction. Suppose the statement holds for n. Then

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 2^{n-1}(1 + x_1 \cdots x_n)$$

and take $x_{n+1} \in [0, 1]$. Then

$$\begin{aligned}(1 + x_1)(1 + x_2) \cdots (1 + x_n)(1 + x_{n+1}) &\leq 2^{n-1}(1 + x_1 \cdots x_n)(1 + x_{n+1}) \\ &\leq 2^{n-1} \cdot 2(1 + x_1 \cdots x_n x_{n+1}) = 2^n(1 + x_1 \cdots x_{n+1})\end{aligned}$$

where we have applied the inequality for $n = 2$ to get the last line and we were justified since $x_1 \cdots x_n \in [0, 1]$. $\square$

Problem 5

Proof. Note that $\sqrt{2}, \sqrt{3}, \sqrt{6} \notin \mathbb{Q}$. Suppose (a, b, c) is a triple of integers with $a + b\sqrt{2} + c\sqrt{3} = 0$. Then we would have

$$(-a)^2 = (b\sqrt{2} + c\sqrt{3})^2 = 2b^2 + 3c^2 + 2bc\sqrt{6}.$$

Now the left hand side is an integer squared and so is also an integer which implies $2b^2 + 3c^2 + 2bc\sqrt{6} \in \mathbb{Z}$. However $2b^2$ and $3c^2$ are also integers so we conclude that $2bc\sqrt{6} \in \mathbb{Z}$. However $\sqrt{6}$ is not rational which implies that $b = 0$ or $c = 0$. In the first case we would have $a + c\sqrt{3} = 0$ which is only satisfied if $a = c = 0$ since $\sqrt{3} \notin \mathbb{Q}$ and in the second case we would have $a + b\sqrt{2} = 0$ which is only satisfied if $a = b = 0$ since $\sqrt{2} \notin \mathbb{Q}$. In either case we conclude that $a = b = c = 0$. $\square$