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Putnam Solutions 2

Problem 3

Proof. Define $a_n = \sqrt{2\pi n}$ and $b_n = \sqrt{(2n+1)\pi}$. We have

$$\int_0^\infty \sin(x)\sin(x^2)dx = \sum_{n=0}^\infty \int_{a_n}^{a_{n+1}} \sin(x)\sin(x^2)dx.$$

Now

$$\begin{aligned}\int_{a_n}^{a_{n+1}} \sin(x)\sin(x^2)dx &= \int_{a_n}^{b_n} \sin(x)\sin(x^2)dx + \int_{b_n}^{a_{n+1}} \sin(x)\sin(x^2)dx \\ &= \int_{2\pi n}^{(2n+1)\pi} \frac{\sin(\sqrt{u})}{2\sqrt{u}}\sin(u)du + \int_{(2n+1)\pi}^{2\pi n+2\pi} \frac{\sin(\sqrt{u})}{2\sqrt{u}}\sin(u)du \\ &= \frac{1}{2}\int_{2\pi n}^{(2n+1)\pi} \left(\frac{\sin(\sqrt{u})}{\sqrt{u}} - \frac{\sin(\sqrt{u+\pi})}{\sqrt{u+\pi}}\right)\sin(u)du.\end{aligned}$$

Now by Taylor's theorem for sufficiently large u

$$\begin{aligned}\sup_{x,y\in[\sqrt{u},\sqrt{u+\pi}]}\left|\frac{\sin(x)}{x} - \frac{\sin(y)}{y}\right| &\le (\sqrt{u+\pi}-\sqrt{u})\sup_{x\in[\sqrt{u},\sqrt{u+\pi}]}\left|\frac{\sin x}{x^2}\right| + \left|\frac{\cos x}{x}\right| \\ &\le (\sqrt{u+\pi}-\sqrt{u})\frac{2}{\sqrt{u}} \\ &= 2\left(\sqrt{1+\frac{\pi}{u}} - 1\right) \\ &\le \frac{2\pi}{u}.\end{aligned}$$

Thus

$$\begin{aligned}&\left|\int_{2\pi n}^{(2n+1)\pi}\left(\frac{\sin(\sqrt{u})}{\sqrt{u}} - \frac{\sin(\sqrt{u+\pi})}{\sqrt{u+\pi}}\right)\sin(u)du\right| \le \\ &\int_{2\pi n}^{(2n+1)\pi}\left|\frac{\sin(\sqrt{u})}{\sqrt{u}} - \frac{\sin(\sqrt{u+\pi})}{\sqrt{u+\pi}}\right|\sin(u)du \\ &\le \int_{2\pi n}^{(2n+1)\pi}\frac{2\pi}{u}\sin(u)du \\ &= 2\pi\int_0^\pi \frac{\sin(u)}{u+2\pi n}du \le \frac{C}{n}.\end{aligned}$$

Now this isn't quite enough to show convergence since we are left with the sum

$$K + \sum_{n=1}^{\infty} \frac{C}{n}$$

but we are nearly there. To fix this do integration by parts on the result from the u-substitution to ensure the exponent of u in the denominator is strictly greater than $\frac{1}{2}$. □

Problem 4

Proof. We have by properties of the logarithm that

$$\begin{aligned}
\sum_{n=2}^{\infty} \log\left(\frac{n^3-1}{n^3+1}\right) &= \sum_{n=2}^{\infty} \log(n-1) - \log(n+1) + \log(n^2+n+1) - \log(n^2-n+1) \\
&= \log(1) + \log(2) + \sum_{n=4}^{\infty} \log(n-1) - \sum_{n=2}^{\infty} \log(n+1) - \log(3) \\
&\quad - \sum_{n=3}^{\infty} \log(n^2-n+1) + \sum_{n=2}^{\infty} \log(n^2+n+1) \\
&= \log\left(\frac{2}{3}\right) + \sum_{n=2}^{\infty} \log((n+2)-1) - \log(n+1) \\
&\quad + \sum_{n=2}^{\infty} \log(n^2+n+1) - \log((n+1)^2-(n+1)+1) \\
&= \log\left(\frac{2}{3}\right).
\end{aligned}$$

□

Problem 6

Proof. Note g is differentiable on $(0, \infty)$ so it suffices to show $g'(x) > 0$. Now

$$\begin{aligned}
\frac{d}{dx} g &= \frac{d}{dx} \frac{f(x)}{x} \\
&= \frac{f'(x)}{x} - \frac{f(x)}{x^2}.
\end{aligned}$$

To establish the claim we therefore need to show that $f(x) < xf'(x)$. Now $f(x) = \int_0^x f'(y)dy$ since f is differentiable, and since f' is strictly increasing we can bound this integral above by $xf'(x)$ which shows $f(x) < xf'(x)$. Therefore we have the claim. □