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Putnam Solutions 5

Problem 1

Proof. We can partition the set of points within distance 1 of a convex n-gon into three disjoint sets: the original polygon, n rectangles which share a side with one of the sides of the n-gon and the other side of length 1, and the slices of a circle of radius 1 which lie between the rectangles. The area of the n-gon is n, the area of the rectangles is $\sum_{\text{edges}} \text{length of edge} \cdot 1 = P$, and we can translate the sectors of the circle to form a whole circle of radius 1 since the polygon is closed. Thus the total area is $A + P + 2\pi$. □

Problem 2

Proof. We define $x' = x/3$ and then A is 3 times the area in the (x', y) plane bounded by the x'-axis, the line $y = 3x'/2$, and the circle $x'^2 + y^2 = 1$ and B is 3 times the area in the (x', y) plane bounded by the y-axis, the line $y = 3mx'$, and the unit circle. For A and B to have the same area the regions in the (x', y) plane must have the same area. Since the regions in the (x', y) plane are mirror images of each other across the line $y = x'$ and we conclude $\frac{2}{3} = 3m$ or $m = \frac{9}{2}$. □

Problem 3

Proof. □

Problem 4

Proof. Without loss of generality we may assume $d_1 < d_2 < \cdots < d_{12}$. Assume there is no triple $i < j < k$ so that d_i, d_j, d_k form the side lengths of an acute triangle. Then $d_i^2 + d_j^2 \leq d_k^2$. We have $d_1^2 > 1$ and $d_2^2 > 1$ so by induction we have $d_i^2 > F_i$ where i is the ith Fibonacci number. Now $F_{12} = 144$ so we must have $d_{12} > 12$, a contradiction. Thus there must exist such a triple. □

Problem 5

Proof. □

Problem 6

Proof. □

Problem 7

Proof. Suppose each pair of faces which share a vertex have a different non-negative integer on them. For a vertex v let a_v be the sum of the values on the faces which contain v. By assumption each of the numbers we add are distinct so $a_v \geq 0 + 1 + 2 + 3 + 4 = 10$. The sum $\sum_v a_v$ is five times the sum of the face values since each vertex is counted five times in this sum, so $\sum_v a_v = 5 \cdot 39 = 195$. However there are 20 vertices so $\sum_v a_v \geq 10 \cdot 20 = 200$ which gives us a contradiction. □

Problem 9

Proof. Consider the tetrahedron with end points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, n)$. The only possibilities for lattice points contained in this tetrahedron are of the form $(0, 0, m)$, $(1, 0, m)$, $(0, 1, m)$, and $(1, 1, m)$ where $m \in \mathbb{N}$. Note no lattice point can be in the interior since the tetrahedron minus its boundary is contained in the box $(0, 1) \times (0, 1) \times (0, n)$ which does not contain any lattice points. Furthermore the only lattice points the three planes which connect $(1, 1, n)$ to a pair of the other vertices can contain are of the form $(0, 0, m)$, $(1, 0, m)$, $(0, 1, m)$, or $(1, 1, m)$. Since these planes are not parallel to the coordinate planes we conclude that the only points of these forms on the boundary of the tetrahedron which is not between $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 1, 0)$ are exactly the vertices. Finally no lattice points other than the vertices can be contained in the boundary of the tetrahedron intersected with the xy-plane. Thus the tetrahedron does not contain any lattice points besides the vertices. The volume of this tetrahedron is $\frac{1}{3}n \times \frac{1}{2} = \frac{n}{6}$ which can be made arbitrarily large by choosing n large. □