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Putnam Solutions 7

Problem 1

Proof.

$$yz(y+z)+xz(x+z)+xy(x+y)=-2$$

□

Problem 2

Proof. We prove the statement by induction. Note the statement is obvious when there are 1, 2, or 3 points. Now suppose M consists of n points and there is a point □

Problem 3

Proof. □

Problem 4

Proof. We prove the statement by induction. It is clearly true for $n=1$ since $1 \leq 1 \leq 2$. If it holds for n then $\sqrt{n} \leq r_n \leq 1+\sqrt{n}$ which implies $1+\frac{n}{r_n} \leq 1+\frac{n}{\sqrt{n}} = 1+\sqrt{n}$ and $1+\frac{n}{r_n} \geq 1+\frac{n}{1+\sqrt{n}} = \frac{1+\sqrt{n}+n}{1+\sqrt{n}} = 1+\sqrt{n}-\frac{\sqrt{n}}{1+\sqrt{n}} = 1+\sqrt{n}-\frac{1}{1+1/\sqrt{n}} \geq 1+\sqrt{n}-1 = \sqrt{n}$. Thus the claim holds for r_{n+1} and statement follows. □

Problem 5

Proof. Without loss of generality we may take $J=[0,1]$ since we can intertwine the continuous map $g: J \to J$ with a continuous map $f:[0,1] \to [0,1]$ by a linear map $\psi: J \to [0,1]$ (ie $g = \psi^{-1} \circ f \circ \psi$). Note since g^n is surjective that g is also surjective: otherwise $g(g^{n-1}([0,1])) \subseteq g([0,1]) \subsetneq [0,1]$ which would imply g^n is not the identity. We also claim g is injective. Suppose there were two distinct point $x_1 \neq x_2$ with $g(x_1)=g(x_2)$. Then $g^n(x_1) = g^{n-1}(g(x_1)) = g^{n-1}(g(x_2)) = g^n(x_2)$ which contradicts g^n being the identity map. Thus g is a homeomorphism from $[0,1] \to [0,1]$ and in particular is monotone increasing or monotone decreasing. First suppose it is monotone increasing. If there is an x such that $x < g(x)$ then $g(x) < g^2(x)$ and so on implying $x < g^n(x) = x$ which is a contradiction. Similarly if there is an x such that $g(x) < x$ then $x = g^n(x) < x$ which also yields a contradiction so we conclude $g(x) = x$ for all x and in particular $g \circ g = I$. Now if g is monotone decreasing then $g \circ g$ is monotone increasing and we again look at the points x with $x \neq g^2(x)$. If $n \equiv 0 \mod 2$ the previous argument gives us that $g \circ g = I$ so assume $n \equiv 1 \mod 2$. Then if $x < g^2(x)$ we have $x < g^{n-1}(x)$ and since g is monotone decreasing we have $x = g^n(x) < g(x)$. This implies $x < g^3(x)$ and so on until we have $x < g^n(x) = x$ a contradiction. The case when $x > g^2(x)$ is identical but with the inequalities flipped. Thus we conclude $g \circ g = I$ and we are done. □

Problem 6

Proof. We prove the statement by induction. The case $n = 1$ is vacuously true since no games are played. Now suppose this statement is true for n players and the number of wins and losses for player r $(1 \leq r \leq n)$ with n players is denoted by w_r and ℓ_r respectively. Now suppose each player plays a new opponent player $n-1$ and call the new number of wins and losses w'_r and ℓ'_r for the original player r and also let w'_{n+1} and ℓ'_{n+1} be the number of wins and losses for the new player. Since the statement for n players holds true even if we relabel the n players by a permutation we may assume without loss of generality that player $n+1$ won against players $1, \cdots, k$ and lost against the last $k+1, \cdots, n$ players. Then we have

$$\begin{aligned}
\sum_{r=1}^{n+1} w'^2_r - \sum_{r=1}^{n+1} \ell'^2_r &= \left(\sum_{r=1}^{n} {w'_r}^2 - {\ell'_r}^2\right) + {w'_{n+1}}^2 - {\ell'_{n+1}}^2 \\
&= \left(\sum_{r=1}^{k} w_r^2 - (\ell_r + 1)^2\right) + \left(\sum_{r=k+1}^{n} (w_r + 1)^2 - \ell_r^2\right) + k^2 - (n-k)^2 \\
&= \left(\sum_{r=1}^{n} w_r^2 - \ell_r^2\right) - \left(\sum_{r=1}^{k} 2\ell_r + 1\right) + \left(\sum_{r=k+1}^{n} 2w_r + 1\right) - n^2 + 2kn.
\end{aligned}$$

Now the first sum is 0 by the inductive hypothesis. Since the total number of losses in the original n player game must be $n(n-1)/2$ (since this is how many games are played) we have $\sum_{r=1}^{k} \ell_r = n^2/2 - n/2 - \sum_{r=k+1}^{n} \ell_r$ which implies the sum of the middle two terms is

$$
\begin{aligned}
-\sum_{r=1}^{k} 2\ell_r + 1 + \sum_{r=k+1}^{n} 2w_r + 1 &= -n^2 + n + (n-k) - k + 2\sum_{r=k+1}^{n} w_r + \ell_r \\
&= -n^2 + 2(n-k) + 2(n-k)(n-1)
\end{aligned}
$$

since the sum of the number of wins and losses of player r is the number of games player r played ie $n-1$. But then the expression for the difference of the sums of the squares of the numbers of wins and losses reduces to

$$-n^2 + 2n(n-k) - n^2 + 2kn = -2n^2 + 2n^2 - 2kn + 2kn = 0.$$

Thus we have established the inductive step and the result follows. □

Problem 7

Proof. □

Problem 9

Proof. □