UT Putnam Practice 1

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2. **For any integer n, let $c(n)$ count the number of ways to express n as a sum of k positive integers a_i, which satisfy $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$. Find a formula to compute $c(n)$ for all n.**

We will show that, for each $1 \leq k \leq n$, there is exactly one way to write n as a sum of k integers with the given constraints, and that for any other k (namely, $k > n$), there are zero such ways.

By the division algorithm, given a fixed n and $k > 0$, there is a unique pair of integers q, r such that $0 \leq r < k$ and $n = qk + r$. We will show a one-to-one correspondence between this representation and the required representation (as a sum) for $1 \leq k \leq n$.

Specifically, if we have $n = qk + r$, then we can write n as the sum

$$n = \underbrace{q + q + \cdots + q}_{k-r} + \underbrace{(q+1) + (q+1) + \cdots + (q+1)}_{r}$$

where we have $k - r$ terms of value q followed by r terms of value $q + 1$. In all, there are $k - r + r = k$ terms, their sum is $q(k - r) + r(q + 1) = kq + r = n$, each successive term is at least the previous, and the last term is at most 1 greater than the first. Note that it may be the case that $r = 0$ (in such a case, there are no $q+1$ terms) but it may *not* be the case that $r \geq k$ (by the division algorithm). Thus, we have shown that there is at least one way to write n as a sum with $1 \leq k \leq n$ terms satisfying the requirements in the claim. Notice that the first term, q, is at least 1 because $k \leq n$ and $r < k$.

Now we must show that there is in fact *only* one such way. To do so, we will use the uniqueness property of the division algorithm. Consider a different sequence with k terms satisfying the required ordering property ($a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$). Since all the terms are in non-decreasing order and the last number cannot be more than one greater than the first, we can let $x = a_1$ and $y = a_1 + 1$, and no terms will be different in value from these two (this follows from the transitivity of the inequality). Then we have

$$n = \underbrace{x + x + \cdots + x + y + y + \cdots + y}_{k}.$$

Noting that the number of y terms (call it r') is at least zero and less than k, we can rewrite this sum as $n = kx + r'$. But this is exactly the representation that is unique, by the division algorithm, so $x = q$ and $r' = r$! Hence, the $n = kq + r$ representation has a one-to-one correspondence with the sum representation, and each $1 \leq k \leq n$ has exactly one sum representation (with the given constraints).

Now, let us consider some $k > n$. The same logic as above does not hold, because we would have $n = kq + r$ where $q = 0$ and $r = n$. (This is in the form of the division algorithm, and so it is the only such representation). Since $q = 0$ and $r < k$, we will have some nonzero number of 0 terms at the beginning of the sum, making it an "invalid" way of writing n.

Therefore, there is exactly one way to write n as a sum given the above constraints with k terms for $1 \leq k \leq n$, and there are zero other ways. Since there are n possible values for k, there are n ways to write the sum in total. So, $c(n) = n$.