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Putnam Prep Week 4, Problem 5

Proof:

Denote the column vectors of A by $\{a_1,\ldots, a_n\}$. Since A has rank r, the column space of A, Col(A), is an r-dimensional subspace of $\mathbb{C}^n$.
So there exists a basis $V = \{v_1, \ldots, v_r\}$ of Col(A). V is a basis for Col(A), so there exists a coordinate vector $[a_i]_V$ of the ith column vector of A with respect to the basis V, for $1 \leq i \leq r$. Let B be the n x r matrix whose ith column vector is v_i, for $1 \leq i \leq r$, and let C be the r x n matrix whose ith column vector is $[a_i]_V$, for $1 \leq i \leq n$. Then a short computation shows that $A = BC$, as required.