Putnam Exam Practice 3, Question 6

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October 2016

1 Problem statement

For each positive integer k let $f(k) = k!/k^k$. Show that for all positive integers m, n we have $f(m+n) < f(m)(f(n)$.

2 Solution

Begin by expanding the function on both sides of the inequality.

$$\frac{(m+n)!}{(m+n)^{(m+n)}} < \frac{m!n!}{m^m n^n} \tag{1}$$

Rearrange to yield:

$$\frac{(m+n)!}{m!n!} < \frac{(m+n)^{(m+n)}}{m^m n^n} \tag{2}$$

Rewriting the left side provides insight on how to attack the right.

$$\binom{m+n}{m} < \frac{(m+n)^{(m+n)}}{m^m n^n} \tag{3}$$

We therefore have

$$\binom{m+n}{m} < \frac{1}{m^m n^n} \sum_{k=0}^{m+n} \binom{m+n}{k} m^k n^{m+n-k} \tag{4}$$

We see that when $k = m$, the powers of m and n cancel, leaving only the binomial coefficient on the left side. All the other terms within the sum are positive, so the entire sum must be strictly greater than the left, showing the inequality to be true.