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Naren Manoj

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Problem 1

Problem 2

Problem 3

Problem 4

Claim: Let the nth Fibonacci number be denoted as F_n, where $F_0 = F_1 = 1$. Then, the desired answer is $\boxed{F_n}$.

Proof: We can proceed by induction. For the sets $\{1\}$ and $\{1, 2\}$, it is obvious that the only minimally selfish sets that can be constructed for each is just $\{1\}$.

Let $S_k = \{1, 2, ..., k\}$ for all integers $k \geq 2$. Let $f(k)$ return the number of minimally selfish subsets of S_k. For our inductive step, assume there exists a positive integer $n - 1$ such that for all $m \leq n - 1$, $f(m) = F_m$. It now suffices to prove that $f(n) = F_n$.

Notice that every minimally selfish subset of S_{n-1} is also a valid minimally selfish subset of S_n. This contributes $f(n - 1)$ counts to the value of $f(n)$. Now, we must determine the number of minimally selfish subsets of S_n that contain n.

Consider the set T_{n-1}, obtained by adding 1 to each of the elements of S_{n-1}. The set of all minimally selfish subsets of S_n containing n must be equal to the set of all minimally selfish subsets of T_{n-1} containing n. To count the latter quantity, if we assume that n is already contained in each of our sets, it remains to correctly populate the sets with the correct remaining elements (some subset of $\{2, 3, ..., n - 1\}$). Notice that doing this is equivalent to counting the number of minimally selfish subsets of $\{1, 2, ..., n - 2\}$, which is simply $f(n - 2)$. Hence, we have $f(n) = f(n - 1) + f(n - 2)$. By our inductive hypothesis, however, we can rewrite $f(n) = F_{n-1} + F_{n-2} = F_n$, as desired. We may thus conclude.

Problem 5

Problem 6

Problem 7

Partition the square into quadrants. By the Pigeonhole principle, it must follow that at least two of the points must lie in the same quadrant. The maximum possible distance between any of these two points occurs when the points are on opposite ends of the same quadrant. This number is therefore half the length of the diagonal of the square, which is $\frac{\sqrt{2}}{2}$, as desired. The same is not necessarily true for a smaller k due to this construction.

Problem 8

Claim: $a(n) = b(n+2) = F_n$, where F_n is the nth Fibonacci number (where $F_0 = F_1 = 1$).

Proof: Notice that if we attempt to recursively define $a(n)$, we can either start our summation sequence with a 1, yielding all possible answers from $a(n-1)$, or we can start the sequence with a 2, yielding all possible answers from $a(n-2)$. It thus follows that $a(n) = a(n-1) + a(n-2)$. Furthermore, trivially, $a(1) = 1$ and $a(2) = 2$. This is the definition of the Fibonacci sequence, and we therefore have $a(n) = F_n$, as desired.

To do the same for b, notice that we can start our sequence with any integer from $2, 3, ..., n$. Hence, $b(n) = \sum_{x=2}^{n} b(n-x)$. Now, we wish to prove that $b(n) = F_{n-2}$ for all integers n. Our base cases are $n = 2$ and $n = 3$. It is obvious that $b(2) = b(3) = 1$, and $F_0 = F_1 = 1$, as desired. For our inductive step, assume that for all integers $k \leq n$, $b(k) = F_{k-2}$. It is a well-known result that the summation of the first n Fibonacci numbers is equal to F_{n+2}; this completes our induction, and we have $b(n+2) = F_n$. It thus follows that $a(n) = b(n+2)$, and we may conclude.

Problem 9

Problem 10

Claim: Let the answer for a given n be $f(n)$. Then, $f(n) = \boxed{\dbinom{n+2}{2}}$.

Proof: Consider (not necessarily distinct) polynomials $A(x)$, $B(x)$, and $C(x)$, all of whose coefficients are either 0 or 1. Then, for some selections of A, B, C, we can rewrite $P(2) = A(2) + B(2) + C(2)$. Notice that, since every integer has a unique binary representation, for any given integer k, there is exactly one polynomial A (and by extension, B and C) that satisfies $A(2) = k$. Hence, the problem reduces to counting the number of ways to write n as $a + b + c$ for some nonnegative integers a, b, c. This is equivalent to counting the number of ways to arrange n boxes and 2 stars, where the value of a is the number of boxes to the left of the first star, the value of b is the number of boxes in between the two stars, and the value of c is equal to the number of boxes to the right of the second star. The number of ways to do this is $\binom{n+2}{2}$, as desired.