Here is a solution to problem A1 of the the 2018 Putnam exam.

We will find three unordered solutions $\{a, b\}$; these obviously give 6 ordered pairs.

For any solution (a, b), let $d = \gcd(a, b)$ and write $a = a'd,\ b = b'd$ so a' and b' are coprime. Further, let $e = \gcd(d, a' + b')$ and write $d = d'e$.

Then $1/a + 1/b = (a + b)/ab = (a' + b')/(a'b'd)$, where the numerator has no common factors with either a' or b', and hence when reduced to lowest terms the fraction will have numerator $(a' + b')/e$ and denominator $a'b'd/e$. Thus we must have

$$a' + b' = 3e \qquad \text{and} \qquad a'b'd' = 2018 = 2 \cdot 1009$$

Since 1009 is prime, the *only* divisors of 2018 are $1, 2, 1009$, and 2018; a', b', and d' must be among these four; in particular, none is a multiple of 3.

Note that $a' \equiv -b' \pmod 3$; without loss of generality let us assume $a' \equiv 1$ and $b' \equiv -1$. Then a' is either 1 or 1009 and b' is either 2 or 2018. In each case, we compute $d' = 2018/(a'b')$; since d' is an integer, one of the four combinations is excluded. We are left with these possibilities:

$$(a', b', d') = (1, 2, 1009), \qquad (1, 2018, 1), \qquad (1009, 2, 1)$$

Correspondingly, we may compute in succession $e = (a' + b')/3,\ d = d'e$, and then $a = a'd$ and $b = b'd$: these will be, respectively

$$(e, d, a, b) = (1, 1009, 1009, 2018), \quad (337, 337, 1009 \cdot 337, 674), \quad (673, 673, 673, 2018 \cdot 673)$$

Thus the possible unordered pairs $\{a, b\}$ are

$$\{1009, 2 \cdot 1009\} \qquad \{674, 337 \cdot 1009\}, \qquad \{673, 1346 \cdot 1009\}$$