Time for Geometry!

The word "geometry" means many different things in mathematics, but most of them are not applicable to the Putnam exam. (I don't think I've ever seen a question really using Differential Geometry, Finite Geometries, etc.) But it is helpful to know some ideas from classical plane and solid geometry — Pythagoras' Theorem, Heron's Theorem, the Platonic Solids, etc. — and to be comfortable with Cartesian geometry, Trigonometry, and the study of curves and surfaces in calculus. Occasionally some combinatorial topology is useful (e.g. Euler's formula for $V - E + F$.) It's also *very* helpful to keep a geometric mindset when pursuing other problems (e.g. to think of sets of equations as describing an algebraic variety, or to view matrices as representing geometric motions). Here is a sampling of geometric problems of many types.

1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

2. Suppose the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfy

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \neq \mathbf{0}$$

Show that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

3. A trapezoid is inscribed in a circle, having a diameter of the circle as its base. A triangle is also inscribed in the same circle, with each of its three sides parallel to a side of the trapezoid. Show that the triangle and the trapezoid have equal area.

4. Suppose four points (x_i, y_i) lie on the hyperbola $xy = 1$, and are also concyclic (i.e. there is a circle containing all four of them). Show that $x_1 x_2 x_3 x_4 = 1$.

5. Find the maximum number of points on a sphere of radius 1 in $\mathbf{R}^n$ such that the distance between any two points is strictly larger than $\sqrt{2}$.

6. A rectangle R is tiled by finitely many rectangles, each of which has at least one side of integral length. Prove that R itself has at least one side of integral length.

7. Suppose S_1 and S_2 are squares contained in the closed unit disk, and each has sides of length 0.9. Show that $S_1 \cap S_2 \neq \emptyset$

8. Every unit square in $\mathbf{R}^2$, when projected onto the x-axis, projects onto a line segment of length at most $\sqrt{2}$. What is the largest possible area of a projection to the x, y plane of a unit cube in $\mathbf{R}^3$?

9. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

10. Can an arc of a parabola inside a circle of radius 1 have length greater than 4?

11. Find the smallest volume bounded by the coordinate planes in $\mathbf{R}^3$ and by a plane which is tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

12. For $i = 1, 2$ let T_i be an acute triangle with side lengths a_i, b_i, c_i and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$. Does it follow that $A_1 \leq A_2$?

13. Show that there are no equilateral triangles in $\mathbf{R}^2$ whose vertices all have integer coordinates.