Abstract Algebra problems from Putnam Exams

If you have never had Math 343K or Math 373K, you're not going to like this set of problems very much. In some of them you are explicitly assumed to understand terms like group, ring, field; subgroup, ideal, quotient; homomorphism, isomorphism; coset, order, identity. However, there are also problems that look more or less approachable without an abstract algebra background, although typically a familiarity with groups and rings will help a lot anyway.

1. Consider a set S and a binary operation $*$ on S. (That is, for each $a, b \in S$, $a * b$ is another element in S.) Assume that $(a*b)*a = b$ for all $a, b \in S$. Prove that $a*(b*a) = b$ for all $a, b \in S$.

2. In the additive group of ordered pairs of integers (m, n) (with addition defined componentwise), consider the subgroup H generated by the three elements

$$(3, 8), \quad (4, -1), \quad \text{and} \quad (5, 4)$$

Then H has another set of generators of the form

$$(1, b), \quad (0, a)$$

for some integers a, b with $a > 0$. Find a.

3. Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S there is an element $z \in S$ such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$ then $a = b$.

4. Let G be a group with operation $*$. Suppose that

(i) G is a subset of $\mathbf{R}^3$ (but $*$ need not be related to any operation on vectors that you may be familiar with).

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ (or both), where $\times$ is the usual cross product in $\mathbf{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for all $\mathbf{a}, \mathbf{b} \in G$.

5. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

6. Let G be a group with identity element e, and let $\phi : G \to G$ be a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = h_1 g_2 h_3 = e$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (that is, $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

7. Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n + 1$.

8. Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $(p^2 1)/2$ distinct nonzero elements of F with the property that for each $\alpha \neq 0$ in F, exactly one of α and $-\alpha$ is in S. Let N be the number of elements in the intersection $S \cap \{2\alpha : \alpha \in S\}$. Prove that N is even.

9. Let F be a finite field having and odd number m of elements. Let $p(x)$ be an irreducible polynomial over F of the form $x^2 + bx + c$ for some $b, c \in F$. For how many elements k of F is $p(x) + k$ irreducible over F?

10. Let R be a ring with the property that if $x \in R$ and $x^2 = 0$, then $x = 0$. Prove that if $x, z \in R$ and $z^2 = z$ then $zxz = xz$.

11. (For something other than abstract algebra ...) For any square matrix A we can define $\sin(A)$ by the usual power series:

$$\sin(A) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin(A) = \begin{pmatrix} 1 & 2014 \\ 0 & 1 \end{pmatrix}.$$

12. Given a set of $n+1$ positive integers, none of which exceeds $2n$, show that one member of the set must divide another member of the set.

13. Show that for every n, the next integer larger than $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

14. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ which satisfy

$$f(x+y) = \frac{f(x) + f(y)}{1 + f(x)f(y)} \qquad \text{for all } x, y \in \mathbf{R}$$