Putnam Practice Problems — Functions and Analysis

1. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that $|f(1)| \leq 1$ and $f(xf(y)) = y\,f(x)$ for all $x, y \in \mathbf{R}$

2. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $f(f(f(f(x)))) = x$ for all $x \in \mathbf{R}$. Show that there is a point $p \in \mathbf{R}$ where $f(p) = p$.

3. Find all continuously-differentiable functions $f : \mathbf{R} \to \mathbf{R}$ for which

$$(f(x))^2 = 2019 + \int_0^x \left((f(t))^2 + (f'(t))^2\right)\,dt$$

for all $x \in \mathbf{R}$.

4. Suppose f and f are non-constant, differentiable functions with $f(0) = 1$ and

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y) \\ g(x+y) &= f(x)g(y) + g(x)f(y) \end{aligned}$$

for all $x, y \in \mathbf{R}$. Show that $(f(x))^2 + (g(x))^2 = 1$ for all $x \in \mathbf{R}$.

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is twice-differentiable and satisfies $f(0) = 0$; $f'(0) > 0$; and $f''(x) \geq f(x)$ for all $x \in \mathbf{R}$. Show that $f(x) > 0$ for all $x > 0$.

6. Suppose $x, y, z \in \mathbf{R}$ are all non-negative and satisfy $x + y + z = 1$. Show that

$$0 \leq xy + yz + zx - 2xyz < \frac{7}{27}$$

7. Suppose x, y, z are the lengths of the sides of a triangle. Show that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$$

Determine when equality holds.

8. Let $n \geq 2$ be a fixed integer. Find the smallest possible constant C such that for all non-negative real numbers $x_1, x_2, \ldots, x_n$ we have

$$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^n x_i\right)^4$$

Determine when equality holds.