Putnam(ish) Problems About Games — Nov 21 2019

1. In the game of NIM a pile of N chips is diminished by two players who alternately remove a number of chips, where the number is chosen from a set M. The person who takes the last chip wins. Determine the values of N for which the first player wins if:

$$(a)\ M = \{1, 2, 4, 8, 16, \ldots\} \qquad\qquad (b)\ \text{(Hard!)}\ M = \{1, 3, 8\}$$

2. (a) Two players play a game in which the first player places a king on an empty 8×8 chessboard, and then, starting with the second player, they alternate moving the king (in accord with the rules of chess) to a square that has not been previously occupied. The player who moves last wins. Which player has a winning strategy?

(b) Suppose a knight is used instead of a king, which player has a winning strategy now?

(c) Suppose a 5×5 chessboard is used instead; answer parts (a) and (b) above.

3. Two players alternately draw diagonals between vertices of a regular polygon. They may connect two vertices if they are non-adjacent (i.e. not a side) and if the diagonal formed does not cross any of the previous diagonals formed. The last player to draw a diagonal wins.

(a) Who wins if the polygon is a pentagon?
(b) Who wins if the polygon is a hexagon?
(c) Who wins if the polygon has a thousand vertices?

4. Two people take turns breaking up a rectangular chocolate bar that is 4×6 squares in size. You can only break the bar along a division between the squares and only in a straight line. So, for example, the first person could break the bar into two 4×3 pieces and the second could break one of the 4×3 pieces into a 4×1 piece and a 4×2 piece. When the bar has been divided into single squares, the person who made the last division wins all the chocolate. Assuming you like chocolate, would you rather go first or second?

5. (1965-B2) In a round-robin tournament with n players $P_1, P_2, \ldots P_n$ (where $n > 1$), each player plays one game with each of the other players and the rules are such that no ties can occur. Let w_r and l_r be the number of games won and lost, respectively, by P_r. Show that

$$\sum_{r=1}^{n} w_r^2 = \sum_{r=1}^{n} l_r^2$$

6. (1989-A4) If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both 1/2. A game is finite if with probability 1 it must end in a finite number of moves.)

7. (1995-B5) A game starts with four heaps of beans, containing 3, 4, 5, and 6 beans respectively. The two players, Agamemnon and Brunhilde move alternately. A move consists of taking either

(a) One bean from a heap, provided at least two beans are left behind in that heap, or

(b) A complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

8. (1997-A2) Players $1, 2, 3, \ldots, n$ are seated around a table and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

9. (2002-B2) This game is played on a polyhedron with at least five faces such that exactly three edges radiate from each of its vertices. Arnold and Betty play the following game. Each player, alternately, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex. Show that Arnold has a winning strategy.

10. (2008-A2). Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?

11. (2009-B2). A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

12. (2011-B4). In a tournament, 2011 players meet 2011 times to play a multiplayer game. Each game is played by all 2011 players together and each ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including $h = k$), if players h and k tied (that is, both won or both lost), then the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that, at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.