We presented quite a few solutions to the number-theory problems. Here are the details of some of the harder ones.

3. Each zero at the end of a number represents another factor of $10 = 2 \cdot 5$, so the number of zeros is the lesser of the exponents on 2 and 5 in the prime factorization of the number. Very well then: how many factors of 2 are there in $n!$? Every other factor in $n! = 1 \cdot 2 \cdot 3 \ldots$ is even, accounting for $\lfloor n/2 \rfloor$ factors of 2. But that's not all: every multiple of 4 contributes an additional factor of 2, and there are $\lfloor n/4 \rfloor$ of these. Adding in the multiples of 8, 16, etc. shows that the exponent on the 2 in the prime factorization of $n!$ is precisely

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \ldots$$

A similar argument works for any prime p and in particular the exponent on the 5 is

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \ldots$$

which is clearly less than the exponent on the 2; hence this number will be the number of zeros in $n!$. In particular, for $n = 1000$ this comes to $200 + 40 + 8 + 1 = 249$ zeros.

Note, by the way, that the exponent on each prime p is less than (but close to)

$$\frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \ldots = \frac{n}{p-1}$$

(summing a geometric progression); for $n = 1000$ and $p = 5$ this would be 250 — just larger than the correct answer of 249.

4. This is kind of a trick (as you should have expected, because there are very few theorems that say "If ... then (something) is prime). Every integer is either a multiple of 3 or one away from a multiple of 3. In the latter case, $p^2 + 2 = (3k \pm 1)^2 + 2 = 3(3k^2 \pm 2k + 1)$ is not prime. So the premises of the problem are met only for $p = 3$, and in that case, yes, $p^3 + 2 = 29$ is indeed prime, and we are done.

11. It is very useful to remember the factorization

$$x^4 + 4y^4 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$$

for this and other problems.

We see that $n^4 + 4^n$ is even (and so, not prime) if n is even. For odd n, note that $n^4 + 4^n = x^4 + 4y^4$ for $x = n$ and $y = 2^{(n-1)/2}$, which is composite as seen in the previous paragraph. (The two factors there are $(x \pm y)^2 + y^2$, which is greater than 1 unless $y = 0$ and $|x| \leq 1$, or $x, y = \pm 1$.)