Here are a few of the solutions that were presented for the geometry questions.

2. We observe that $\mathbf{a} \times (\mathbf{a}+\mathbf{b}+\mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a} = \mathbf{0}$ by hypothesis, which means $\mathbf{a}+\mathbf{b}+\mathbf{c}$ is a scalar multiple $\mathbf{a}$ (which is a nonzero vector because $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$). Likewise $\mathbf{a}+\mathbf{b}+\mathbf{c}$ is a scalar multiple $\mathbf{b}$ which is also not only nonzero but linearly independent of $\mathbf{a}$ (again because $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$). So $\mathbf{a}+\mathbf{b}+\mathbf{c}$ must be the zero vector.

4. If $(x, y)$ is a point which lies both on the circle $(x-h)^2 + (y-k)^2 = r^2$ and on the hyperbola $xy = 1$ then we have $(x^2 - hx)^2 + (1 - kx)^2 = r^2 x^2$, that is, $x$ is a root of the quartic polynomial $x^4 - (2h)x^3 + (h^2 + k^2 - r^2)x^2 - (2k)x + 1 = 0$. So the four points which lie in the intersection give us the four roots of this quartic, whose (sum is $2h$ and ... and) product is $+1$.

6. Suppose the big rectangle is the Cartesian product $[0, b] \times [0, d]$. On this rectangle we define the function $f(x, y) = cos(2\pi x)\cos(2\pi y)$ and we observe that the integral of $f$ over the whole rectangle is $\sin(2\pi b)\sin(2\pi d)/4\pi^2$. On the other hand, the integral is the sum of the integrals over the smaller rectangles $R_i = [a_i, b_i] \times [c_i, d_i]$. We may evaluate the integral over $R_i$ in the same way as

$$(\sin(2\pi b_i) - \sin(2\pi a_i))(\sin(2\pi d_i) - \sin(2\pi d_i))/4\pi^2.$$

but if the side length $b_i - a_i$ is an integer, then the first factor is zero, and if the other side length $d_i - c_i$ is an integer, then the second factor is zero. Therefore the hypotheses of the problem assure that the integral over every $R_i$ is zero, and hence the integral over the tiled union is zero too. So either $b$ or $d$ must be an integer, meaning the big rectangle has a side of integral length.

7. If a square of side-length $s$ is placed inside the unit circle, and $s$ is large enough, then the square will enclose the center of the circle. To see this, rotate the figure so that edges are aligned with the axes. If the top edge is above the center of the circle, push the square lower until a bottom vertex touches the circle. Note that the square can go lower still until the second bottom vertex also touches the circle. At that point we have made the top edge as low as possible, and the square touches at pointe $(\pm s/2, y)$ where $y = -\sqrt{1 - (s/2)^2}$; the other vertices are at $(\pm s/2, y+s)$. Thus the center of the circle is still inside the square if $y + s > 0$, i.e. if $s^2 > 1 - (s/2)^2$. The cut-off value of $s$ works out to be $2/\sqrt{5} = (2/5)\sqrt{5}$ which is about 0.894.

So any squares with side-length 0.9 must all contain the center, and in particular cannot be disjoint.

8. This may be a well-known result, but I didn't know it so I have to prove it:

Lemma: If $S$ is a subset of a plane $P$ and $S'$ is its orthogonal projection onto another plane $P'$ then $\text{area}(S') = |\cos(\theta)|\text{area}(S)$ where $\theta$ is the angle between $P$ and $P'$.

Proof: It will suffice to prove this for the unit square in $P$. Then the result will follow for other squares in $P$ by scaling and translation; then for shapes with rectangular boundaries by taking unions; then for all measurable sets by taking limits.

So let $\mathbf{i}$ and $\mathbf{j}$ be an orthonormal basis in $P$, and let $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, a unit normal vector to $P$. Let $\mathbf{n}$ be a unit normal vector for $P'$; then $\cos(\theta) = \mathbf{n} \cdot \mathbf{k}$. The projection function $f : P \to P'$ sends every vector $v$ to a vector $v - \lambda n$ where $\lambda$ is chosen to make this vector perpendicular to $\mathbf{n}$, so we compute $f(v) = v - (v \cdot \mathbf{n})\mathbf{n}$. Then the projection of the unit square is the parallelogram spanned by $f(\mathbf{i})$ and $f(\mathbf{j})$; that parallelogram has area $||f(\mathbf{i}) \times f(\mathbf{j})||$. So let us compute this. First write $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, so that $a^2 + b^2 + c^2 = 1$. In this notation we have $f(\mathbf{i}) = \mathbf{i} - a\mathbf{n}$ and $f(\mathbf{j}) = \mathbf{j} - b\mathbf{n}$, so $f\mathbf{i}) \times f(\mathbf{j}) = \mathbf{k} - b(\mathbf{i} \times \mathbf{n}) + a(\mathbf{j} \times \mathbf{n})$. Now, $\mathbf{i} \times \mathbf{n} = b\mathbf{k} - c\mathbf{j}$ and $\mathbf{j} \times \mathbf{n} = -a\mathbf{k} + c\mathbf{i}$ so altogether we have $f\mathbf{i}) \times f(\mathbf{j}) = ac\mathbf{i} + bc\mathbf{j} + (1 - a^2 - b^2)\mathbf{k} = c\mathbf{n}$, which is a vector of magnitude $|c|$. So the square of area 1 projects to a parallelogram of area $|c| = |\mathbf{k} \cdot \mathbf{n}|$, as claimed.

Apply this argument now to all the sides of the unit cube in $\mathbf{R}^3$, with $P'$ being a plane with any unit normal $\mathbf{n}$. The dot products of $\mathbf{n}$ with the normals of the sides will be $a$ or $b$ or $c$ (each one for two of the six sides) so the sum of the areas of the projections will be $2(|a| + |b| + |c|)$. The projection covers its image in precisely a 2-to-1 fashion, so the area of the whole projected cube is $|a| + |b| + |c|$.

That leaves only the question of how to maximize this function over all the unit vectors $\mathbf{n}$. Clearly this asks that we find the point on the sphere (in the first octant, say) which lies on the furthest of the planes $x + y + z =$constant, and that will be the point on the sphere which lies on the line $x = y = z$. That's the point $\mathbf{n} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, and at this point the objective function takes the value of $3/\sqrt{3} = \sqrt{3}$.

In short, the largest shadow of a cube is the one that projects along a body diagonal, leaving a hexagon of area $\sqrt{3}$.

10. Consider the single parabola $y = x^2$ and the circles $x^2 + y^2 = Ky$ centered above the vertex of that parabola, which also pass through that vertex. For each such $K > 1$ we will get two points of intersection. If the points of intersection are $(\pm L, L^2)$ then $L^2 + (L^2)^2 = K(L^2)$ forces $K = L^2 + 1$. The length of that curve is given in Calculus as being $2\int_0^L \sqrt{1 + 4x^2}\, dx$. Using calculus-2 techniques, I make out a formula for $F$ to be

$$F(L) = L\sqrt{1 + 4L^2} + \frac{1}{2}\ln(2L + \sqrt{1 + 4L^2})$$

The first term is more than $2L^2$, and the second term is larger than $(1/2)\ln(4L)$. So the whole integral shows that the arclength exceeds $2(L^2 + 1)$ when $\ln(4L) > 4$, i.e. for $L > e^4/4$ (about 13.65). For these $L$, then, the parabola is longer than twice the diameter of the enclosing circle. Scale the figure down to discover that there is an appropriately long parabola in the *unit* circle.

(It turns out the the parabola is longer for all $L > 8.275$ or so. The maximum ratio of excess is near $L = 13.681467$ but even there, the parabola is only longer by a factor of about 1.00066757!)

12. Allow me to change notation: we have a triangle $T$ of area $A$ and having sides $x_1, x_2, x_3$ and another triangle $T'$ of area $A'$ and sides $x_i' \le x_i$. Consider the three ratios $r_i = x_i'/x_i$, each at most one. Let $r$ be the largest of three; renumber the sides so that $r = r_3$, say. Then consider a third triangle $T''$ obtained from $T$ by contracting by a factor of $r$: $x_i'' = rx_i$

and $A'' = r^2 A \leq A$. By choice of $r$ we have $x'_i \leq x''_i$ for $i = 1, 2, 3$ but actually $x'_3 = x''_3$. We will show $A' \leq A''$, which we have already shown is at most $A$, so we will be done.

Very well, rotate the triangles so the the edge $E$ of length $x'_3 = x''_3$ is horizontal and the triangles $T'$ and $T''$ lie above this line segment, reaching up to vertices $V'$ and $V''$ respectively. The fact that $x'_1 \leq x''_1$ tells us that $V'$ lies in a disk centered at one end of $E$ and having radius $x''_1$. Likewise the inequality $x''_2 \leq x'_2$ puts $V'$ into a disk centered at the other end of $E$. It follows that $V'$ is in the top half of a lune (the intersection of two disks), where the top-most point of the lune is the top vertex $V''$ of triangle $T''$. But that means $V'$ is at a lower height than $V''$. Since the triangles have the same bottom edge $E$, the triangle with a lower height is the one with a lower area. Thus $A' \leq A''$. We have already noted $A'' \leq A$ so we are done.

13. Pick's Theorem!