Nov 7 2019: Putnam Exam: meet Linear Algebra!

Here are some of the solutions I can sketch quickly.

1. No such matrices exist: since $Tr(AB) = Tr(BA)$, the trace on the left is zero but the trace on the right is not.

2. The given identity is equivalent to $(A - I)(B - I) = I$, i.e. it asserts that $A - I$ and $B - I$ are inverses of each other. But then they are also inverses in the opposite order, which untangles to say that $A + B$ is *also* equal to BA.

3. The stated conditions assert that the block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} D^t & B^t \\ -C^t & -A^t \end{pmatrix}$$

are inverses of each other. Again, reverse the order of the product.

4.

5. These are called "circulant matrices", and it's easy to see that the vector

$$(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$$

is an eigenvector if $\zeta^n = 1$. Taking ζ to be any of the n distinct nth roots of unity then gives us n linearly independent eigenvectors, so that these are the only eigenvectors (up to scalar multiplication of course). The corresponding eigenvalue is $\sum M_i \zeta^{i-1}$.

6.

7. The fact that A annihilates a nonzero matrix B merely tells us that A has a nontrivial kernel (i.e. it has a rank less than 3). Then, think about the possible forms of the Jordan Canonical Form for a 3×3 matrix, knowing that one of the eigenvalues is 0. There's a small collection of types, and for each one of them it's not hard to pick out a matrix D whose entries are all zeros and 1s, and which annihilates A on both sides.

8.

9.

10. Taking $y = 0, 1, 2$ on the left gives three linearly independent polynomials, but on the right only gives polynomials in the span of $a(x)$ and $b(x)$, which is a 2-dimensional space. So no such a, b, c, d can exist.