Combinatorics Problem 4

William Blount
Putnam Exam Practice

November 1, 2019

Problem 4. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1,...,x_n)$ of integers such that $|x_1|+...+|x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

Proof. Consider the function $g(m,n,k)$ which denotes the number of n-tuples $(x_1,...,x_n)$ of integers such that $|x_1|+...+|x_n| \leq m$ and where exactly k of the x_i are non-zero. First note that there are $\binom{n}{k}$ ways to place the non-zero elements in the tuple. Each of the non-zero terms can either be positive of negative. Therefore there are 2^k ways to choose the signs for the non-zero terms. Finally we count the number of ways to choose k non-zero natural numbers such that their sum is less than or equal to m. We do this by considering the partitioning of m "1"s into $k+1$ partitions. The first k partitions must have at least one "1" while the last partition may have any number of "1"s. This is equivalent to counting the number of ways to partition $m+1$ "1"s into $k+1$ partitions with at least one "1" in each partition. This in turn is equivalent to counting the number of ways to partition $m+1-(k+1)$ "1"s into $k+1$ partitions with any number of "1"s in each partition. By the usual stars and bars argument this is

$$\binom{(m+1-(k+1))+(k+1)-1}{(k+1)-1} = \binom{m}{k}$$

Therefore

$$g(m,n,k) = \binom{n}{k} \cdot 2^k \cdot \binom{n}{k} = 2^k \binom{m}{k}\binom{n}{k}$$

Furthermore

$$g(m,n,k) = 2^k \binom{m}{k}\binom{n}{k} = 2^k \binom{n}{k}\binom{m}{k} = g(n,m,k)$$

Summing $g(m,n,k)$ over all k gives us $f(m,n)$ so it follows that

$$f(m,n) = \sum_{k\in\mathbb{Z}} g(m,n,k) = \sum_{k\in\mathbb{Z}} g(n,m,k) = f(n,m)$$

as desired. □