PUTNAM COMBO SOLS

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1. Problem 1

Theorem 1.1.
$$\binom{pa}{pb} \equiv \binom{a}{b} \mod p$$

Proof. Let S be the set of permutations of the set $A = (X, X, X, ..., Y, Y, Y, ...)$ where A has pb X's and pa elements. Note that $|S| = \binom{pa}{pb}$.

Define a function $P : S \to \mathbb{N}^a$ where $\forall S_0 \in S$
$$P(S_0) = [x_1, x_2, \ldots, x_a]$$
where x_k is the amount of X's in $\{S_0(i)\}_{k(p-1)<i\leq kp}$

Note that $\forall i \leq a, 0 \leq x_i \leq p$ and $\sum_{i=1}^{a} x_i = pb$.

Consider the set $B = \{0, p\}^a$. $B \subseteq \mathbb{N}^a$. As a result, we can consider $P^{-1}(B)$.

$P^{-1}(B)$ consists of all of the permutations of A in which the X's are grouped into b groups of p and where all the Y's are grouped into $b - a$ groups of p. As a result, $P^{-1}(B)$ is equivalent to permuting b large X groups and $b - a$ large Y groups, so $|P^{-1}(B)| = \binom{a}{b}$.

Now consider $P^{-1}(\mathbb{N}^a - B)$.
$$\forall\, C \in \mathbb{N}^a - B, \exists\, k \in \mathbb{N}$$
$$0 \leq C[k] \leq p$$
Since $\forall\, S_0 \in P^{-1}(C)$ elements $k(p-1)+1$ to kp can be permuted independently of the others,
$$\binom{k}{p} |(|P^{-1}(C)|)$$
$p|\binom{k}{p}$, so $p|(|P^{-1}(C)|)$
$$\binom{pa}{pb} = |S| \equiv |P^{-1}(\mathbb{N}^a)| \equiv |P^{-1}(B)| + |P^{-1}(\mathbb{N}^a - B)| \equiv |P^{-1}(B)| = \binom{a}{b} \mod p$$
□

Question 1.2. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3 by 3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3 by 3 matrix is completed with five 1s and four 0s. Player 0 wins if the determinant is 0 and Player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

Theorem 1.3. *Player 0 can force a win.*

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Proof. I demonstrated in the session that this game is equivalent to a game in which P0 tries to prevent P1 from getting 3 1's in a row or a column of a matrix. I will demonstrate that P0 can successfully do so.

Some terminology,

an OR/OC (open row/column) is a row/column which contains nothing but a single 1.

a DOR/DOC (double open row/column) is a row/column which contains nothing but two 1's.

In order for P1 to win, there must be a DOR/DOC after P0's turn. P0 can play such that at the end of his turn $\#OR + \#OC \leq 1$ and $\#DOR = \#DOC = 0$.

I will prove so by induction.

Base case:

After P0's 0th turn, there are 0 OR/OC and 0 DOR/DOC.

Inductive step:

After P0's kth turn, there is (WLOG) at most one OR and no OCs (for one OC and no ORs just flip every row and column).

After P1's (k+1)th turn, there are three possibilities.

1. $\#OR + \#OC \leq 2$ and $\#DOR = \#DOC = 0$

P0 can block one of the OR/OCs.

2. $\#OR = 2$ and $\#OC = 1$ and $\#DOR = \#DOC = 0$

Since there are two ORs, there must be at least one OR that intersects with the OC at an open square (there can only be one 1 on the OC). P0 places a 0 there and blocks one OR and one OC.

3. $\#DOR = 1$ and $\#OC \leq 1$ and $\#DOC = 0$

P0 blocks the DOR.

As a result, P0 can always make it so that there are no DOR or DOC after his turn, making it impossible for P1 to win.

□