Problem 3:

Let $*$ be a commutative and associative binary operation on a set S. Assume that $\forall x, y \in S$, $\exists z \in S$ such that $x * z = y$ (z may depend on x and y). Show that if $a, b, c \in S$ and $a * c = b * c$, then $a = b$.

Proof. Since $\forall x, y \in S$, $\exists z \in S$ such that $x * z = y$, we can say $\exists z_a, z_b \in S$ such that $(a * c) * z_a = a$ and $(a * c) * z_b = b$. Further, because $a * c = b * c$, we can say $(b * c) * z_a = a$ and $(b * c) * z_b = b$. Also, since $a * c = b * c$, we can say $(a * c) * z_a * c * z_b = (b * c) * z_a * c * z_b$. Using the commutative and associative properties of the $*$ binary operation, we can manipulate this equality in the following way.

$$(a * c) * z_a * c * z_b = (b * c) * z_a * c * z_b$$

$$((a * c) * z_b) * c * z_a = ((b * c) * z_a) * c * z_b$$

$$b * c * z_a = a * c * z_b$$

$$(b * c) * z_a = (a * c) * z_b$$

$$a = b$$

Thus, $a = b$.

□