Putnam Practice: Geometry

4. Show that for any set of five points on a sphere there is a set of four of them that lie on a closed hemisphere

For any sphere we can define its center as the origin (0, 0, 0). By taking any plane through the origin we can split this sphere into hemispheres based on what side of the plane the hemisphere is on.

If we assume that the five points $\{p_1, p_2, p_3, p_4, p_5\}$ on the sphere are not all collinear, we can define a plane P as the span of two of these points treated as vectors. Let us call these points that define the plane p_1 and p_2. Right now, plane P splits the sphere into two closed hemispheres that both contain p_1 and p_2. Let us call these hemispheres h_1 and h_2. The rest of the points p_3, p_4, p_5 must be in at least one of h_1 or h_2. We can see both hemispheres already contain two points, so no matter how we split up the rest of the points at least one of the hemispheres will get at least two of them, so it will contain a set of four points.

If all the of the five points are collinear, we can define a plane P as the span of one of the points and any other vector that is linearly independent. This plane will contain all five points, so both hemispheres will contain five points, which is greater than four. Therefore, there will always be a closed hemisphere that contains at least four of the five points.