Connor Colombe
Putnam Prep
Assignment 1
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(Equilateral triangle with lattice vertices) Prove that there does not exist an equilateral triangle with lattice point vertices in the plane.

Solution:
Suppose to the contrary there exists an equilateral triangle with integer coefficients in the plane. We will find its area in two different ways. First we use **Pick's Theorem** which states:

Theorem 1 (Picks Theorem in 2-D). *Let P be a simple polygon with integer coordinates. Let I and B be the number of lattice points on the interior and on the boundary of P respectively. Then we can express the area of P as*

$$A = I + \frac{B}{2} - 1$$

.

From Pick's Theorem, it follows that since I and B are integers, our triangle will have rational area.

Now we will compute the area of our triangle using the standard area formula for an equilateral triangle with side length s

$$A = \frac{\sqrt{3}s^2}{4}.$$

Since our first calculation revealed that $A \in \mathbb{Q}$. It follows that $\frac{\sqrt{3}s^2}{4}$ must be rational. However, we will now show that $\frac{\sqrt{3}s^2}{4}$ is irrational.

To see why, consider any two vertices our our triangle and label them (x_1, y_1) and (x_2, y_2) where $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. The squared distance between these points is s^2, and using the distance formula we get

$$s^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \in \mathbb{Z}.$$

If s^2 is an integer then this implies that $\frac{x^2}{4}$ is rational and thus $\sqrt{3}$ is rational which is a contradiction.[1] Therefore, there does not exist an equilateral triangle with integer coordinates.

[1] This can be shown using the classic irrationality proof technique.