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Q.3) Given the positive numbers a and b and the natural number n, find the greatest among the n + 1 monomials in the binomial expansion of $(a+b)^n$

Solution:

Let's first make a substitution that will reduce the number of components we need worry about. WLOG let $a \leq b$, and let $r = \frac{b}{a}$ we can then write our equation as

$$\begin{aligned}(a+b)^n &= a^n(1+\frac{b}{a})^n \\ &= a^n(1+r)^n\end{aligned}$$

This a^n term will be present in every term in the expansion and therefore will not affect the relative size each monomial. We have now reduced the problem to one of fewer variables. We wish to find the monomial with largest value in the expansion of $(1+r)^n$ such that $r \geq 1$.

One way to compare the relative sizes of adjacent terms in the expansion is to take their ratio. Let R_k be defined as the ratio of the $k+1$ term in the expansion compared to the k-th term. It follows

$$\begin{aligned}R_k &= \frac{\binom{n}{k+1}r^{k+1}}{\binom{n}{k}r^k} \\ &= r\left(\frac{n-k}{k+1}\right)\end{aligned}$$

Now we have a way to determine the relative size of adjacent terms in the expansion but how can we use this to find the largest? Observe that if we take the derivative with respect to k, we find

$$\frac{dR_k}{dk} = -r\frac{n+1}{(k+1)^2} < 0.$$

That is, this ratio is decreasing as we move along the sequence! This implies that index for the largest term in the sequence k is the first for which $R_k < 1$. We now set up the inequality $R_k \geq 1$ and solve for k.

$$k \geq \frac{rn-1}{r+1}$$

We want the smallest k for which this inequality fails to hold. Therefore the desired index is

$$\boxed{k = \left\lfloor \frac{rn - 1}{r + 1} \right\rfloor + 1}$$

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In summary, given the expression $(a + b)^n$ for positive a, b and natural number n, if we define r to be the ratio of the larger to the smaller of a, b then the largest term of the monomial expansion will occur at the k-th index where $k = \left\lfloor \frac{rn-1}{r+1} \right\rfloor + 1$.