Connor Colombe
Putnam Seminar

Q.8) A function f satisfies the equation

$$f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x$$

for every real number x except for $x = 0$ and $x = 1$. Find a closed formula for f.

Solution:

A common strategy for these sorts of functional equations is to look for symmetry and try to choose input values that make the equations "nicer". The term that seems to be the issue is the $1 - \frac{1}{x}$ as an input. However, maybe it has some hidden nice property! Lets write this as $\frac{x-1}{x}$ and plug it into itself a few times. We see

$$\left.\frac{x-1}{x}\right|_{x=\frac{x-1}{x}} = \frac{1}{1-x}$$
$$\left.\frac{1}{1-x}\right|_{x=\frac{x-1}{x}} = x.$$

That is, if we take $\frac{x-1}{x}$ and plug it into itself twice, we get back x! Using this we can rewrite our original functional equation in the following ways

$$f(x) + f\left(\frac{1-x}{x}\right) = 1 + x$$
$$f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1-x}{x}$$
$$f\left(\frac{1}{1-x}\right) + f(x) = 1 + \frac{1}{1-x}$$

But this is a system of three equations with three unknowns! Substituting $f\left(\frac{1}{1-x}\right)$ and then $f\left(\frac{x-1}{x}\right)$ to solve for $f(x)$, we arrive at

$$\boxed{f(x) = \frac{1}{2}\left(1 + x + \frac{1}{1-x} - \frac{x-1}{x}\right)}$$

This function satisfies our original functional equation and is defined for all real x except for 0 and 1.