Putnam Study Group 2020-2021, week 1

Today's problems are a grab-bag of questions that use tools from different areas of mathematics, and are of varying levels of difficulty. Find some that you like and work up a solution that you can share with the other participants!

(1) Friendship is a symmetric and non-reflexive relation. That is, if Alice is Bob's friend, then Bob is Alice's friend, and nobody can count themselves as a friend. Is it possible to have an isolated community of 30 people such that no two people have the same number of friends?

(2) (a) A 10×10 matrix A is made from 92 1's and 8 0's. What is the largest possible value of $\det(A)$?

(b) What are the possible values of $\det(A)$ if A is made from 91 1's and 9 0's?

(3) Given the positive numbers a and b and the natural number n, find the greatest among the $n+1$ monomials in the binomial expansion of $(a+b)^n$.

(4) For this problem, assume that all years have 365 days, and that February 29 doesn't exist. Also, note that December 31 and January 1 are consecutive days. What's the largest set of people such that no two have the same birthday and

(a) no two birthdays fall on consecutive days?

(b) no two birthdays are exactly a week apart?

(c) no two birthdays are exactly 5 days apart?

(5) Charlotte writes a test consisting of 100 questions, where the answer to each question is either TRUE or FALSE. Charlottes teacher announces that for every five consecutive questions on the test, the answers to exactly three of them are TRUE. Just before the test starts, the teacher whispers to Charlotte that the answers to the first and last questions are both FALSE.

(a) Determine the number of questions for which the correct answer is TRUE.

(b) What is the correct answer to the sixth question on the test?

(c) Explain how Charlotte can correctly answer all 100 questions on the test.

(6) A cow is tethered to a post that is exactly at the corner of a 10m-by-10m shed on level ground.

(a) What is the area of the pasture in which she can graze, if her tether is exactly 20m long?

(b) What if the tether has length $L > 20$?

(c) What if, in part (a), we assume the cow is grazing on a spherical planet of radius 10km ?

(7) Let C be a bounded convex region of the plane. Show that there is a line that divides both the area and perimeter of C in half.

(8) A function f satisfies the equation

$$f\left(x\right) + f\left(1 - \frac{1}{x}\right) = 1 + x$$

for every real number x except for $x = 0$ and $x = 1$. Find a closed formula for f.

(9) Prove there is an integer k for which $k^3 - 36k^2 + 51k - 97$ is a multiple of 3^{2015}.

(10) Show that for every $n > 2$ there is a set of n points in the plane meeting these two conditions:

(i) For any two of the points, the line segment between them has irrational length.

(ii) For any three of the points, the triangle between them has rational area.