Connor Colombe
Putnam Seminar

Q.1) Evaluate
$$\int_0^{12} \frac{x^2}{x^2 + (12 - x)^2}\,dx$$

Solution:

We will solve a more general problem to highlight a useful integration technique. Suppose you have the integral

$$I = \int_a^{b-a} \frac{f(x)}{f(x) + f(b - x)}\,dx \tag{1}$$

where $b \geq 2a$. Let's make the substitution $u = b - x$. The integral now becomes

$$I = \int_{b-a}^{a} -\frac{f(b - u)}{f(b - u) + f(u)}\,du$$
$$= \int_a^{b-a} \frac{f(b - u)}{f(b - u) + f(u)}\,du$$
$$= \int_a^{b-a} \frac{f(b - x)}{f(b - x) + f(x)}\,dx \tag{2}$$

If we now sum equations (1) and (2), we get

$$2I = \int_a^{b-a} \frac{f(x)}{f(x) + f(b - x)}\,dx + \int_a^{b-a} \frac{f(b - x)}{f(b - x) + f(x)}\,dx$$
$$= \int_a^{b-a} \frac{f(x) + f(b - x)}{f(x) + f(b - x)}\,dx$$
$$= \int_a^{b-a} dx$$
$$= b - 2a$$

Isolating for I, we find

$$\boxed{I = \frac{b - 2a}{2}}.$$

Next, note that integral in the problem statement has the same structure as the integral in equation (1) with $a = 0$ and $b = 12$. Using our derived formula, we find

$$\boxed{\int_0^{12} \frac{x^2}{x^2 + (12 - x)^2}\,dx = 6}$$