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Q.2) Assuming $a > 0$ and $b > 0$, evaluate

$$\lim_{n\to\infty} n^{b-a}\frac{1^a + 2^a + \cdots + n^a}{1^b + 2^b + \cdots + n^b}$$

Solution:

Let this limit be denoted L. This looks like it might be a Riemann sum in disguise! Recall the limit definition of a definite integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=0}^{n} f\left(a + \frac{(b-a)k}{n}\right). \tag{1}$$

We will try to manipulate L to get it into this form. First we factor our n^a and n^b from the numerator and denominator respectively.

$$\begin{aligned}
L &= \lim_{n\to\infty} n^{b-a}\frac{n^a}{n^b}\frac{\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \cdots + \left(\frac{n}{n}\right)^a}{\left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \cdots + \left(\frac{n}{n}\right)^b} \\
&= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \cdots + \left(\frac{n}{n}\right)^a}{\left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \cdots + \left(\frac{n}{n}\right)^b} \\
&= \lim_{n\to\infty} \frac{\left(\frac{0}{n}\right)^a + \left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \cdots + \left(\frac{n}{n}\right)^a}{\left(\frac{0}{n}\right)^b + \left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \cdots + \left(\frac{n}{n}\right)^b} && \text{(add zero to top and bottom)} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}\left(\left(\frac{0}{n}\right)^a + \left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \cdots + \left(\frac{n}{n}\right)^a\right)}{\frac{1}{n}\left(\left(\frac{0}{n}\right)^b + \left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \cdots + \left(\frac{n}{n}\right)^b\right)} && \text{(multiply top and bottom by } 1/n\text{)} \\
&= \frac{\lim_{n\to\infty} \frac{1}{n}\left(\left(\frac{0}{n}\right)^a + \left(\frac{1}{n}\right)^a + \left(\frac{2}{n}\right)^a + \cdots + \left(\frac{n}{n}\right)^a\right)}{\lim_{n\to\infty} \frac{1}{n}\left(\left(\frac{0}{n}\right)^b + \left(\frac{1}{n}\right)^b + \left(\frac{2}{n}\right)^b + \cdots + \left(\frac{n}{n}\right)^b\right)}
\end{aligned}$$

Note that in the last step, we used the quotient limit property since the denominator sum is nonzero. Now, notice that the numerator and denominator both have the structure present in equation (1). Substituting in the equivalent integrals, we find

$$\begin{aligned} L &= \frac{\int_0^1 x^a dx}{\int_0^1 x^b dx} \\ &= \frac{\frac{1}{a+1}}{\frac{1}{b+1}} \\ &= \boxed{\frac{b+1}{a+1}} \end{aligned}$$