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Q.3) Let $f(x) = \frac{1}{4+x^4} + A$ where A is constant, and let $F(x)$ be an antiderivative of f. Find a value of A for which F has exactly one critical point.

Solution:

Since $F(x) = \int f(x)dx$, if we want to solve for the critical points of $F(x)$, we want to solve the equation

$$\frac{d}{dx}F(x) = 0.$$

However, by the fundamental theorem of calculus, this reduces to

$$f(x) = 0.$$

The problem is now reduced to finding A such that $f(x)$ has only one real root. Since $4 + x^4 > 0$ for all real x, we can solve this directly.

$$\begin{aligned} \frac{1}{4+x^4} + A &= 0 \\ 1 + A(4 + x^4) &= 0 \\ (1 + 4A) + 4x^4 &= 0 \end{aligned} \tag{1}$$

This is an equation of the form $x^4 + c$ for some constant c. The only value of c for which a quartic of this form has single real solution is $c = 0$. This implies that $\boxed{A = -\frac{1}{4}}$ is the value of A for which F has exactly one critical point.