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Putnam Seminar

Q.4) Let $f(n) = \sum_{k=2}^{\infty} \frac{1}{k^n \cdot k!}$. Compute $\sum_{n=2}^{\infty} f(n)$

Solution:

Let S denote the sum we would like to calculate. We can write S as

$$S = \sum_{n=2}^{\infty} \left(\sum_{k=2}^{\infty} \frac{1}{k^n \cdot k!} \right) \tag{1}$$

If we switch the order of summation, we can do the following

$$\begin{aligned}
S &= \sum_{k=2}^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{k^n \cdot k!} \right) && (2)\\
&= \sum_{k=2}^{\infty} \frac{1}{k!} \left(\sum_{n=2}^{\infty} \frac{1}{k^n} \right) && (3)\\
&= \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{1}{1-\frac{1}{k}} - 1 - \frac{1}{k} \right) && (4)\\
&= \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{k}{k-1} - \frac{k+1}{k} \right) && (5)\\
&= \sum_{k=2}^{\infty} \frac{1}{(k-1)! \cdot (k-1)} - \sum_{k=2}^{\infty} \frac{k+1}{k! \cdot k} && (6)\\
&= \sum_{k=2}^{\infty} \frac{1}{(k-1)! \cdot (k-1)} - \sum_{k=2}^{\infty} \frac{1}{k! \cdot k} - \sum_{k=2}^{\infty} \frac{1}{k!} && (7)\\
&= \left(\sum_{k=1}^{\infty} \frac{1}{k! \cdot k} - \sum_{k=2}^{\infty} \frac{1}{k! \cdot k} \right) - \sum_{k=2}^{\infty} \frac{1}{k!} && (8)\\
&= 1 - \left(\sum_{k=0}^{\infty} \frac{1}{k!} - 1 - 1 \right) && (9)\\
&= \boxed{3 - e} && (10)
\end{aligned}$$

Some important notes to make are that we can expand (3) using a convergent geometric series since the ratio $|\frac{1}{k}| < 1$ for $k \geq 2$, but we must account for the two missing terms for $n = 1$ and $n = 0$. In (7) we re-index our first sum so that it will telescope with the second sum. Lastly, in (8) we re-index the final sum so that we will get the Taylor expansion of e^x at $x = 0$.