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Putnam Seminar

Q.5) Show that this improper integral converges:

$$\int_0^\infty \sin(x)\sin(x^2)dx$$

Solution: If we find a function $f(x)$ such that $f(x) \geq \sin(x)\sin(x^2)$ for all $x \geq 0$ and $\int_0^\infty f(x)dx < \infty$, then by the comparison test for improper integrals[1], we are done.

We notice that $|\sin(x)| \leq 1$ so it might be worth trying to evaluate the integral $\int_0^\infty \sin(x^2)dx$. It seems there is no straightforward way to proceed but recall $\sin(t) = \text{Im}[e^{it}]$. Using this we can write our new integral as

$$\begin{aligned} I &= \int_0^\infty \sin(x^2)dx \\ &= \int_0^\infty \text{Im}[e^{ix^2}]dx \\ &= \text{Im}\left[\int_0^\infty e^{ix^2}dx\right] \end{aligned} \tag{1}$$

Next, observe that (1) looks very similar to a Gaussian integral! We can achieve a Gaussian integral if we notice $i = -1/i$. Our integral now becomes

$$I = \text{Im}\left[\int_0^\infty e^{-(1/i)x^2}dx\right]. \tag{2}$$

But this is a half of a Gaussian integral of the form $\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\pi/a}$ which can be evaluated using the standard trick for Gaussian integrals[2]. Substituting half of this value with $a = 1/i$ for our integral in (2), we find

$$I = \text{Im}[\frac{1}{2}\sqrt{\pi i}]. \tag{3}$$

To finish, we must find the imaginary part of $\sqrt{i}$. If we write i using the complex exponential $\sqrt{i} = \sqrt{e^{i\pi/2}} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$. Therefore, the imaginary part and thus the value of (3) and thus the value of I is $\boxed{\sqrt{\frac{\pi}{8}}}$. Since I has finite value and the integrand is always greater than the desired function, the integral $\int_0^\infty \sin(x)\sin(x^2)dx$ converges.

[1] see https://tutorial.math.lamar.edu/classes/calcii/ImproperIntegralsCompTest.aspx
[2] see https://en.wikipedia.org/wiki/Gaussian_integral