Connor Colombe
Putnam Seminar

Q.7) Compute
$$\int_0^{\frac{\pi}{2}} \frac{dx}{\left(\sqrt{\sin(x)}+\sqrt{\cos(x)}\right)^4}$$

Solution:
While this integral looks intimidating (in my opinion at least!) it can be solved by iterated u-substitution. In its current form, we don't have much to work with. Let's try factoring to see if we can make it more manageable. Let the value of our integral be denoted I.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{dx}{\left(\sqrt{\sin(x)}+\sqrt{\cos(x)}\right)^4} \\
&= \int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2(x)\left(\sqrt{\tan(x)}+1\right)^4} && \text{(factor out the } \sqrt{\cos x}\text{)} \\
&= \int_0^{\frac{\pi}{2}} \frac{\sec^2(x)dx}{\left(\sqrt{\tan(x)}+1\right)^4} && \text{(rewrite as } \sec x\text{)}
\end{aligned}$$

Now recall that $d/dx(\tan x) = \sec^2 x$. Motivated by this, let $u = \tan(x)$. Making the substitution and adjusting the limits we find

$$I = \int_0^{\infty} \frac{du}{\left(\sqrt{u}+1\right)^4}. \tag{1}$$

Wanting to get rid of the square root term, we then make the substitution $y^2 = u$. The integral expressed in terms of y is now

$$I = \int_0^{\infty} \frac{2y}{\left(y+1\right)^4}dy. \tag{2}$$

Lastly, we would like for the denominator to be a single term. We can achieve this through the substitution $t = y + 1$. Once more, if we rewrite the integral in terms of of the new variable t, we get

$$I = 2\int_1^{\infty} \frac{t-1}{t^4}dt. \tag{3}$$

Which we can now separate this into two elementary integrals and solve,

$$\begin{aligned} I &= 2\int_1^\infty \frac{t-1}{t^4}dt \\ &= 2\left(\int_1^\infty \frac{t}{t^3}dt - \int_1^\infty \frac{t}{t^4}dt\right) \\ &= 2\left(\frac{1}{2} - \frac{1}{3}\right) \\ &= \boxed{\frac{1}{3}} \end{aligned}$$