Putnam Practice: Number Theory

7. Let \( f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4) \) where \( a, b, c, d, e \) are integers and \( a \neq 0 \). Show that if \( r_1 + r_2 \) is a rational number and if \( r_1 + r_2 \neq r_3 + r_4 \) then \( r_1r_2 \) is a rational number too.

If you multiply out \( a(z - r_1)(z - r_2)(z - r_3)(z - r_4) \) and look at the constant term, you can see that \( e = ar_1r_2r_3r_4 \), which is an integer. Similarly, if you focus on the \( z^3 \) term, you can see that \( b = -a(r_1 + r_2 + r_3 + r_4) \), which is also an integer.

By rearranging, you can see that \( -\frac{b}{a} - (r_1 + r_2) = (r_3 + r_4) \). Notice that \( b, a, (r_1 + r_2) \) are all rational, so \( -\frac{b}{a} - (r_1 + r_2) = (r_3 + r_4) \) must be rational too.

To save space, let \( R = (r_1 + r_2) \), so \( R \) is a rational number. By plugging in \( R \) into \( f(z) \), you get \( f(R) \), which must be rational too, as a rational number raised to an integer power is still rational, and the sum of rational numbers is rational.

\[
f(R) = a(R - r_1)(R - r_2)(R - r_3)(R - r_4) = ar_1r_2(R^2 - R(r_3 + r_4) + r_3r_4)
\]

By plugging in \( R \) into \( f(z) \), you get \( f(R) \), which must be rational too, as a rational number raised to an integer power is still rational, and the sum of rational numbers is rational.

\[
f(R) = ar_1r_2(R^2 - R(r_3 + r_4)) + ar_1r_2r_3r_4
\]

Recall \( ar_1r_2r_3r_4 = e \).

\( f(R), a, R, (r_3 + r_4), e \) are all rational and adding or multiplying rational numbers still preserves their rationality, so \( r_1r_2 \) is rational too.

Note that for this equation to provide useful information about \( r_1r_2 \), we must avoid indeterminate forms (when the denominator is zero). \( a(R^2 - R(r_3 + r_4)) \neq 0 \), so \( R^2 \neq R(r_3 + r_4) \), and \( (r_1 + r_2) \neq (r_3 + r_4) \), which is good because that’s what the problem says, so we can exclude that case from our solution. However, this equation also does not provide information for when \( (r_1 + r_2) = 0 \), so we must test that case separately.

If we multiply out \( a(z - r_1)(z - r_2)(z - r_3)(z - r_4) \) again and focus on the \( z \) term, we see that \( d = -a(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4) = -ar_1r_2r_3r_4 \left(\frac{1}{r_1} + \frac{1}{r_2}\right) - ar_1r_2(r_3 + r_4) \). If \( r_1 + r_2 = 0 \) and \( r_1, r_2 \neq 0 \), then \( \frac{1}{r_1} + \frac{1}{r_2} = 0 \) too. (If \( r_1, r_2 = 0 \) then \( r_1r_2 = 0 \), which is rational).

Our equation simplifies to \( d = -ar_1r_2(r_3 + r_4) \). We have seen that \( d, a, (r_3 + r_4) \) are rational so if \( (r_1 + r_2) \neq (r_3 + r_4) \), \( r_1r_2 \) is still rational even if \( (r_1 + r_2) = 0 \).

Putting it together, we can see that if \( r_1 + r_2 \) is rational, even if it equals zero, and if \( (r_1 + r_2) \neq (r_3 + r_4) \), then \( r_1r_2 \) is rational.