Putnam Practice: Number Theory

7. Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers and a $\neq$ 0. Show that if $r_1 + r_2$ is a rational number and if $r_1 + r_2 \neq r_3 + r_4$ then $r_1 r_2$ is a rational number too.

If you multiply out $a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ and look at the constant term, you can see that $e = ar_1 r_2 r_3 r_4$, which is an integer. Similarly, if you focus on the z^3 term, you can see that $b = -a(r_1 + r_2 + r_3 + r_4)$, which is also an integer.

By rearranging, you can see that $-\frac{b}{a} - (r_1 + r_2) = (r_3 + r_4)$. Notice that $b, a, (r_1 + r_2)$ are all rational, so $-\frac{b}{a} - (r_1 + r_2) = (r_3 + r_4)$ must be rational too.

To save space, let $R = (r_1 + r_2)$, so R is a rational number. By plugging in R into $f(z)$, you get $f(R)$, which must be rational too, as a rational number raised to an integer power is still rational, and the sum of rational numbers is rational.

$$f(R) = a(R - r_1)(R - r_2)(R - r_3)(R - r_4) = ar_1 r_2 (R^2 - R(r_3 + r_4) + r_3 r_4)$$

$$f(R) = ar_1 r_2 \big(R^2 - R(r_3 + r_4)\big) + ar_1 r_2 r_3 r_4$$

$$\frac{f(R) - ar_1 r_2 r_3 r_4}{a\big(R^2 - R(r_3 + r_4)\big)} = r_1 r_2$$

Recall $ar_1 r_2 r_3 r_4 = e$.

$f(R), a, R, (r_3 + r_4), e$ are all rational and adding or multiplying rational numbers still preserves their rationality, so $r_1 r_2$ is rational too.

Note that for this equation to provide useful information about $r_1 r_2$, we must avoid indeterminate forms (when the denominator is zero). $a\big(R^2 - R(r_3 + r_4)\big) \neq 0$, so $R^2 \neq R(r_3 + r_4)$, and $(r_1 + r_2) \neq (r_3 + r_4)$, which is good because that's what the problem says, so we can exclude that case from our solution. However, this equation also does not provide information for when $(r_1 + r_2) = 0$, so we must test that case separately.

If we multiply out $a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ again and focus on the z term, we see that $d = -a(r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4) = -ar_1 r_2 r_3 r_4 \left(\frac{1}{r_1} + \frac{1}{r_2}\right) - ar_1 r_2 (r_3 + r_4)$. If $(r_1 + r_2) = 0$ and $r_1, r_2 \neq 0$, then $\frac{1}{r_1} + \frac{1}{r_2} = 0$ too. (If $r_1, r_2 = 0$ then $r_1 r_2 = 0$, which is rational). Our equation simplifies to $d = -ar_1 r_2 (r_3 + r_4)$. We have seen that $d, a, (r_3 + r_4)$ are rational so if $(r_1 + r_2) \neq (r_3 + r_4)$, $r_1 r_2$ is still rational even if $(r_1 + r_2) = 0$.

Putting it together, we can see that if $r_1 + r_2$ is rational, even if it equals zero, and if $(r_1 + r_2) \neq (r_3 + r_4)$, then $r_1 r_2$ is rational.