1) Pf: Let prime p be of n bits, then 2^{n-1} <p< 2^n.

$$2k\ C\ k\ =\ [(2^n)!]\ /\ [(2^{n-1})!\ (2^{n-1})!] =\ [(2^{n-1}+1)(2^{n-1}+2)...(2^n)]\ /\ (2^{n-1})!$$

Notice that p is contained in the numerator but not in the denominator since p is a prime larger than 2^{n-1} and less than 2^n. Therefore, p| 2kCk. QED

3) show that every rational number…. Quotient of prime!...

Pf: We first prove that all primes p_n can be written as products/ quotients of some factorials of primes. (*)
Base case: p1= 2 = 2!/1! So * holds for n=1
n=k. Assume * holds for all prime numbers up to p_k.
n=k+1. Observe p_{k+1} = (p_{k+1})! / (p_{k+1} -1)!. p_{k+1} is less than p_{k+1}, so it can be written as the product of primes smaller than p_{k+1} -- or smaller than or equal to p_k. (if a prime factor of p_{k+1} - 1 is larger than or equal to p_{k+1} then p_{k+1}-1 is larger than or equal to p_{k+1}, a contradiction.)
Since by induction hypothesis, all primes from p_1 to p_k can be written as products/quotients of prime factorials, the product of these primes (for example, p_{k+1} -1) also can be written as products/ quotients of prime factors. Thus, p_{k+1} can be written as a product/quotient of prime factorials. * holds for n=k+1.
Therefore, * holds for all prime numbers.
All rational numbers can be written as the quotient of two integers, and all integers can be factored into prime numbers. All rational numbers are therefore the products/ quotients of prime numbers, and since * is true for all prime, all rational numbers can be written as products/ quotients of prime factorials. QED

7) Let f(z) =az4+bz3+cz2+dz+e=a(z−r1)(z−r2)(z−r3)(z−r4) where a,b,c,d,eare integers and a6= 0. Show that if r1+r2is a rational number and if r1+r2 is not r3+r4 then r1r2 is a rational number too.

Pf
Divide both sides of the equation by a, we get

$$z^4 + \alpha z^3 + \beta z^2 + \gamma z + \varepsilon = (z - r1)(z - r2)(z - r3)(z - r4)$$

Since a, b, c, d, e are integers, $\alpha,\ \beta,\ \gamma,\ \varepsilon$ are rational numbers, where

$$\varepsilon\ =\ r1 * r2 * r3 * r4 \qquad (1)$$

$$\gamma = -(r1 * r2 * r3 + r1 * r2 * r4 + r1 * r3 * r4 + r2 * r3 * r4) = (r1\ r2)\ (r3 + r4)\ +\ (r3\ r4)\ (r1 + r2) \qquad (2)$$

$$\beta = r1\ r2\ + r1r3 + ...r3\ r4 =\ r1r2 + r3\ r4\ +\ (r1 + r2)\ (r3 + r4) \qquad (3)$$

$$\alpha = -(r1\ +\ r2\ +\ r3 +\ r4) \qquad (4)$$

From (4), since α and r1+r2 are both rational, r3+r4 is also rational. So (r1+r2)(r3+r4) is also rational. From (4), β is rational, so r1 r2 + r3 r4 is rational, call it q1. Multiplying both sides by (r1+r2), we get

$q1\ (r1 + r2)\ =\ r1\ r2\ (r1 + r2) + r3\ r4\ (r1 + r2)$ (5)

(2)-(5) and rearrange, we get $[\gamma - q1\ (r1 + r2)]\ /\ (r1 + r2 - r3 - r4)\ = r1\ r2$

γ , q1, r1+r2, and (r1+r2-r3-r4) are all rational and r1+r2 -(r3+r4) is not 0 bc r1+r2 does not equal r3+r4, r1 r2 is rational too. QED

6)

Pf by induction:

base case: m=n=1, $(1+1)!/(1+1)^{1+1}$ = ½ < $1!/1^1 * 1!/1^1$ = 1. The claim holds.

Induction step: let the claim work for m=a and n=b. That is,

$$\frac{(a+b)!}{(a+b)^{a+b}} < \frac{a!}{a^a}\,\frac{b!}{b^b}$$

Now we add one to either m or n. WLOG, consider the case when m=a+1, n=b.

$$\frac{(a+b+1)!}{(a+b+1)^{a+b+1}} < \frac{(a+b+1)(a+b)!}{(a+b+1)(a+b)^{a+b}} < \frac{(a+b)!}{(a+b)^{a+b}} < \frac{a!}{a^a}\,\frac{b!}{b^b} \text{ ?????}$$