Connor Colombe
Putnam Seminar

Q.6) Show that every positive integer n is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another

Solution: We will proceed by strong induction on n.

Base Case(s):

- $n = 1$. We can write $1 = 2^0 3^0$
- $n = 2$. This is already in the desired format.

Inductive Hypothesis: Assume that we can write every positive integer less than n in the desired fashion. We wish to show we can also write n in this manner. We will consider two cases.

Case 1: If n is even. We write $n = 2k$ for some positive integer k. Note that by our inductive hypothesis, we can write

$$k = \sum_{i=1} 2^{\alpha_i} 3^{\beta_i}$$

where $2^{\alpha_i} 3^{\beta_i} \nmid 2^{\alpha_j} 3^{\beta_j}$ for all $i \neq j$. Note that if a pair of terms in this expansion for k is indivisible, then multiplying each by 2 will not effect their divisibility. Therefore we have

$$n = 2k = \sum_{i=1} 2^{\alpha_i + 1} 3^{\beta_i}$$

where no summand divides another.

Case 2: Now suppose n is odd. If $n = 3^k$ for some positive integer k then we are done. Otherwise, Let 3^k be the largest power of 3 less than n. Consider $m = n - 3^k$. It must be the case that m is even and by our inductive hypothesis it must be the case

$$m = \sum_{i=1} 2^{\alpha_i} 3^{\beta_i}$$

where $2^{\alpha_i} 3^{\beta_i} \nmid 2^{\alpha_j} 3^{\beta_j}$ for all $i \neq j$. Note that since m is even, there cannot be a term of the form $2^0 3^\beta$ in the summand for m, since all terms of the form $2^r 3^s$ are even for $r \geq 1$. This implies that 3^k will not be divisible by any of the terms in the summand for m. Could there be a term in the summand that is divisible by 3^k? Since $3^k < n < 3^{k+1}$, the only possible divisible term would be $2^1 3^k$. However if this term was present in the summand for m, then

$$\begin{aligned} n &= 3^k + m \\ &\geq 3^k + 2^1 3^k \\ &= 3^{k+1}. \end{aligned}$$

This contradicts our earlier assumption that $n < 3^{k+1}$. Therefore we can write

$$\begin{aligned} n &= 3^k + m \\ &= 3^k + \sum_{i=1} 2^{\alpha_i} 3^{\beta_i} \end{aligned}$$

Where no two terms in the summand divide one another, as desired.