Connor Colombe
Putnam Seminar

Q.3) Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$$

Solution: Note that any positive rational number r can be expressed as a fraction $\frac{p}{q}$ where p and q are positive integers that relatively prime. If we can show that that we can express p and q each as quotients of products of factorials of primes, then their quotient and thus r can be expressed in this manner as well. Therefore, it is sufficient to prove that every non-negative integer N can be expressed as quotients of products of factorials of primes. To do so, we will first prove that we can do so for just the prime numbers using strong induction. Then, it follows from the fundamental theorem of arithmetic that we can achieve a desired expression for every positive integer. We now proceed with proving this holds for all prime numbers.

Base Case(s):
$N = 2$: We can write $2 = \frac{2!2!}{2!}$

$N = 3$: We can write $3 = \frac{3!}{2!}$

Inductive Hypothesis: Now assume that we can express all positive prime integers as quotients of products of factorials of primes for all primes $q \leq N$. Consider the positive first prime integer $p > N$.

Since p is prime, then we can express it as

$$p = \frac{p!}{(p-1)!}.$$

Now by the fundamental theorem of arithmetic, $(p-1)!$ can be factored into a unique product of primes and each prime in the product is strictly less than p. Therefore, by our inductive hypothesis, we can write $(p-1)!$ as a quotient of a product of factorials of primes and thus p in this manner.

Since we can write every prime as a quotient of products of prime factorials and every positive integer can be factored into a product of primes we can write every positive integer in this manner. It follows from the logic in the first paragraph that we can express every positive rational number in this way.