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Putnam Seminar

Q.6) Let m, n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

Solution:

Let's start by isolating the terms with factorials and the terms with exponents to see if we can make sense of this inequality. We want to show

$$\frac{(m+n)!}{m! \cdot n!} < \frac{(m+n)^{m+n}}{n^n \cdot m^m} \tag{1}$$

We notice that $\frac{(m+n)!}{m! \cdot n!} = \binom{m+n}{n}$. Substituting this into (1) we get the new expression we wish to prove

$$\binom{m+n}{n} < \frac{(m+n)^{m+n}}{n^n \cdot m^m},$$

or equivalently

$$\binom{m+n}{n} n^n \cdot m^m < (m+n)^{m+n}. \tag{2}$$

Now recall the general form of the binomial expansion for integers a, b, r

$$(a+b)^r = \sum_{k=0}^{n} \binom{r}{k} a^k b^{r-k}$$

For $a, b > 0$, each term in the expansion is positive and thus for $r > 1$ it follows $\binom{r}{k} a^k b^{r-k} < (a+b)^r$ for all values of $k = 0, 1, \ldots, r$. The expression we wish to prove (2) follows from plugging in the values $a = n, b = m, k = n$ and $r = m + n > 1$ for positive integers n and m.