NUMBER THEORY!!! (Oct 14, 2020)

1. An *n-bit prime* is a prime number which uses n bits when written in binary. Show that every n-bit prime number divides the central binomial coefficient $C_k = \binom{2k}{k}$ where $k = 2^{n-1}$. (For example, 5 is a 3-bit prime so it should divide $C_4 = \binom{8}{4}$; indeed $C_4 = 70$ is a multiple of 5, as well as the other 3-bit prime, which is 7.)

For extra credit devise a quick method to compute these central binomial coefficients. (It would then be possible to find all the prime divisors of a number N by computing the gcd of N and C_{2^n} for $n = 1, 2, \ldots$).

2. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

3. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$$

4. Let h and k be positive integers. Prove that for every $\epsilon > 0$, there are positive integers m and n such that

$$\epsilon < |h\sqrt{m} - k\sqrt{n}| < 2\epsilon$$

5. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2 - 1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3 - 1)$ are distinct modulo p^3.

6. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

7. Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers and $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and if $r_1 + r_2 \neq r_3 + r_4$ then $r_1 r_2$ is a rational number too.

8. Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001$$