Putnam week 4

8) [This is really jank but here's a scenario; not mathematical but i'm just having fun with this.]

Left side: n students are staging a show that requires 2 acts. For any 2 students, they have an act. To organize the performance, they pick 2 acts out of nC2 acts.
Right side: To organize the performance, they can either pick 3 students and have one of them perform twice, leading to 3 possibilities (for students A, B, C, the possible performances are AB, AC; AB, BC; AC, BC). Or, they can pick 4 students and have everyone perform exactly once (for students A, B, C, D, the possible performances are AB, CD; AC, BD; AD, BC). In total, there are

$$3*nC3 + 3*nC4 = 3* (n+1)C4$$

Since both sides are doing the same thing, they are equal to each other. QED

2)

Notice that each vertex is connected to 5 faces. If at every vertex, the five faces that share the same vertex all have different numbers (face numbers), the sum at each vertex is at least 0+1+2+3+4=10. Now we add up the sums at the 12 vertices. The final sum should be at least 10*12 = 120. Notice that because each face has 3 vertices, each face number is counted exactly 3 times, which means the sum of all face numbers is at least 120/3 = 40. However, the actual sum is 39, a contradiction. Thus, it's not possible that at all vertices, the five face numbers are different. In other words, there are at least two faces that share a vertex and have the same integer written on them. QED