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Putnam Seminar

Q.1) Show that if p is prime then for all positive integers a and b,

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$$

Solution:

Consider the left hand side, on its face. It looks like the number of ways to pick a committee of size pb from a pool of pa people. However, we will count this is a different way. First, we split the pa people into equal groups of size a and assign each a different color shirt $C_1, C_2, \ldots, C_p$. Our problem has been reduced to counting the number of ways to pick a grouping of size pb from the union of all of these different color groups. We can then come up with two distinct and exhaustive methods to pick a committee. In one method, we pick exactly b people from each group to form a committee, and the other where we do not. We shall call the committees of the former structure Type 1 and latter Type 2.

If we want to count the number of Type 1 committees, we need to pick exactly b members from each color group. There are $\binom{a}{b}$ ways to get members from C_1, $\binom{a}{b}$ ways to get members from C_2, and so on. Therefore there are $\binom{a}{b}^p$ ways to pick a group of size pb by picking exactly b members from each color group. Since p is a prime, Fermat's little theorem tells us

$$\binom{a}{b}^p \equiv \binom{a}{b} \pmod{p}.$$

Now, if we can show the number of Type 2 committees is a multiple of p then we will have achieved our result.

Note that if not all color groups contribute an equal number of members to the committee, then we can place a non-trivial ordering on the number of committee members selected from each group. Let $A_1, A_2, \ldots, A_p$ be the number of members selected from each color group in a particular selection. WLOG suppose that $A_1 \geq A_2 \geq \ldots \geq A_p$ and not all are equal. Suppose there are K total different committees that can be formed that satisfy this inequality with not all equal A_i. By symmetry, there are $p!$ distinct orderings of $A_1, A_2, \ldots, A_p$, each with K corresponding possible committees, therefore the number of possible committees of the second type is $p! \cdot K$ which is a multiple of p.

In summary, we have shown that the left hand side can be interpreted as picking a committee of size pb from p groups of size a, each group with a distinct color shirt. We partitioned the number of possible committees into two types: one where each color shirt group has an equal number of members and ones that do not. The former is congruent to $\binom{a}{b}$ mod p and the latter is a multiple of p. The desired equivalence follows.